ALGEBRA

Question 1

An arithmetic sequence is defined in such a way that the 15th term, $u_{15} = 92$, and the 3rd term, $u_3 = 56$.

(a) Determine
  (i) the common difference.
  (ii) the first term.

(b) Calculate S_{12}.

Question 2

A geometric sequence is such that its 4th term, $u_4 = 135$ and the ratio $\frac{u_9}{u_4} = 3^5$.

(a) Find (i) the common ratio.
  (ii) the first term.

(b) If $S_{10} = a(b-1)$, determine the real constants a and b.

Question 3

Consider the sequence defined by the equation $u_n = 1 - 9n, n = 1, 2, 3, \ldots$, where u_n represents the nth term of the sequence.

(a) Write down the value of u_1, u_2 and u_3.

(b) Calculate $\sum_{n=1}^{10}(1-9n)$.

Question 4

Solve for x in each of the following equations

(a) $\log_2 x^4 = \log_2 16$.

(b) $\log_x 27 = 3$.

(c) $\log_4 32 = x$.

(d) $\log_3(1-x) - \log_3 x = 2$.

Question 5

The nth term of a sequence is given by

$$u_n = \frac{2}{3} \times 3^n, n = 1, 2, 3, \ldots .$$

If $\sum_{n=1}^{20} \left(\frac{2}{3} \times 3^n \right) = a(3^{20} - b)$, determine the values of a and b.

Question 6

Solve the following equations for x.

(a) $x - 4 = 4$.
(b) $|x - 4| = 4$.
(c) $\log_2(x - 4) = 4$.
(d) $x^2 - 4 = 4$.
(e) $\log_2(x - 2) + \log_2(x + 2) = 2$.

Question 7

Solve the following equations for x.

(a) (i) $2x - 1 = 14$.
    (ii) $\frac{3 - x}{2} = x$.
(b) $(x - 2)(x + 1) = 1$.

Question 8

(a) (i) Factorise the quadratic expression $x^2 - 6x + 5$.
    (ii) Solve the inequality for x, where $x^2 - 6x + 5 > 0$.
(b) For what values of x is
    (i) $\log_{10}(6 - x)$ defined?
    (ii) $\log_{10}(x - 4)$ defined?
(c) Solve for x, the equation $\log_{10}(x^2 - 6x + 5) = \log_{10}(6 - x) + \log_{10}(x - 4)$.

Question 9

(a) (i) Expand $(a+1)(a^2+2a-1)$.

(ii) Solve for a the equation $a^3+3a^2+a-1 = 0$.

(b) Solve for x where $(\ln x)^3+3(\ln x)^2+\ln x = 1$.

Question 10

(a) (i) Expand $(x+a)(x+1)$.

(ii) Factorise x^2+3x+2.

(b) Solve each of the following equations for x,

(i) $x^2-(e+1)x+e = 0$.

(ii) $e^{2x}-(e+1)e^x+e = 0$.

Question 11

The first three consecutive terms of an arithmetic sequence are $k-2$, $2k+1$ and $4k+2$.

(a) Find the value of k.

(b) Find (i) u_{10}.

(ii) S_{10}.

Question 12

The first three terms of a geometric sequence are given by $\sqrt{3}+1$, x and $\sqrt{3}-1$, where $x > 0$.

(a) Find x.

(b) Find S_∞.

Question 13

Given that $\frac{1}{2}+1+2+2^2+\ldots+2^{10} = a \times 2^b + c$, find the values a, b and c.

Question 14

A geometric sequence is such that each term is equal to the sum of the two terms directly following it. Find the positive common ratio.

Question 15

An arithmetic sequence is such that $u_p = q$ and $u_q = p$.

(a) Find, in terms of p and q, u_1.

(b) Find S_{p+q}.

Question 16

Solve each of the following equations for x.

(a) $x^2 + x - 2 = 0$.

(b) $(\ln(x))^2 - \ln\left(\frac{1}{x}\right) - 2 = 0$.

(c) $2^x - 2 \times 2^{-x} + 1 = 0$.

Question 17

(a) Expand $(2a + 1)(3 - a)$.

(b) Solve (i) $\log_{10}(2x - 5) + \log_{10}(x) = \log_{10}3$.

(ii) $2 \times 3^{2x-1} - 5 \times 3^{x-1} - 1 = 0$.

Question 18

Find k if (i) $\log_2(2k - a) = 3$.

(ii) $\log_{10}k = 1 + \log_{10}5$.

(iii) $3\ln 2 - 2\ln 3 = -\ln k$.

Question 19

(a) If $\log(a - b) = \log a - \log b$, find an expression for a in terms of b, stating any restrictions on b.

(b) If $a^2 + b^2 - 2ab = 4$ and $ab = 4$, where $a > 0$, $b > 0$, find the value of $2\log_{20}(a + b)$.

Question 20

Solve the simultaneous equations $\begin{aligned} 9^{x-y} &= 27 \\ 2^{x+y} &= 32 \end{aligned}$.

Question 21

If $x = \log_a n$ and $y = \log_c n$ where $a, c > 0$ and $n \neq 1$, prove that

$$\frac{x-y}{x+y} = \frac{\log_b c - \log_b a}{\log_b c - \log_b a},\ b > 0.$$

Question 22

A sequence of numbers have the property that x, 12, y, where $x > 0$, $y > 0$ form a geometric sequence while 12, x, $3y$ form an arithmetic sequence.

(a) If $xy = k$, find k.

(b) Find the value of x and y.

(c) For the sequence x, 12, y, find (i) the common ratio.
(ii) the sum to infinity.

Question 23

The numbers α and β are such that their arithmetic mean is 17 while their geometric mean is 8. Find α and β.

Question 24

Given that the sequence with general term u_i, has its sum defined by $\sum_{i=1}^{n} u_i = n(n+2)$, find

(a) $\sum_{i=1}^{10} u_i$.

(b) (i) the first term, u_1.
(ii) the second term, u_2.

(c) (i) the rth term.
(ii) the 20th term.

Question 24

(a) Given that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = a + \frac{b}{n+1}$, find a and b.

(b) Find $\sum_{k=10}^{99} \frac{1}{k(k+1)}$, giving your answer in the form $\frac{m}{n}$, where m and n have no common factors.

Question 25

(a) Show that $\log_{(a^n)}(x^n) = \log_a x$, $a > 0$, $a \neq 1$, $x > 0$, $n \neq 0$.

(b) Hence find $\log_{\sqrt{5}} 5$.

Question 26

One solution to the equation $5^{x+1} = 3^{x^2-1}$ is given by $x = a + \log_3 b$. Find the other solution and the values a and b.

Question 27

(a) Simplify $\frac{5^{2n-1} - 5^{2n-3}}{5^{2n} + 5^{2n-2}}$.

(b) If $2^x = 5$ and $2 = 5^y$, find xy.

Question 28

Solve the equation $\log_4(3 \times 2^{x+1} - 8) = x$.

Question 29

Prove by mathematical induction, that $8^n - 3^n$ where $n \in \mathbb{Z}^+$, is divisible by 5.

Question 30

Find all values of x such that $2^{2x} - 2^{x+1} < 3$.

Question 31

Find the constant term in the expansion $\left(2x + \frac{1}{x}\right)^{2n}$.

Question 32

Three consecutive coefficients in the expansion of $(1 + x)^n$ are $\binom{n}{r}$, $\binom{n}{r+1}$ and $\binom{n}{r+2}$ and are in the ratio 6 : 3 : 1.

(a) Show that $2n - 3r = 1$ and $3n - 4r = 5$.

(b) Which terms are they?

Question 33

Given that $(3 - i)$ is a zero of the polynomial $p(x) = x^3 - 5x^2 + 4x + 10$, find the other zeros.

Question 34

Given that $(\sqrt{3} + i)^8 = -a^7(b + ci)$, find the values of a, b and c.

Question 35

Express $\frac{1 + i\sqrt{3}}{1 - i\sqrt{3}}$ in the form $z = re^{i\theta}$.

Question 36

If $\frac{1}{(1+i)^2} + \frac{1}{(1-i)^2} = a + bi$, $a, b \in \mathbb{R}$, find a and b.

Question 37

Given that $w = \frac{z}{z - i}$, where $z = a + bi$, $a, b \in \mathbb{R}$, show that if w is a real number then z is a pure imaginary number.

Question 38

If $m + ni = \dfrac{i(4-3i)}{(1-i)^2}$,

(a) find m and n.

(b) express $\dfrac{i(4-3i)}{(1-i)^2}$ in the form $z = re^{i\theta}$.

Question 39

Given that $w = -1 + \sqrt{3}i$ and $z = 1 - i$, find

(a) (i) $|w|$, $\text{Arg}(w)$.
(ii) $|z|$, $\text{Arg}(z)$.
(iii) $|zw|$, $\text{Arg}(zw)$.

(b) (i) Find wz.

(ii) Hence find the exact value of $\sin\left(\dfrac{5\pi}{12}\right)$.

Question 40

If $z = \dfrac{a}{1+i}$ and $w = \dfrac{b}{1+2i}$ where $a, b \in \mathbb{R}$, find a and b given that $z + w = 1$.

Question 41

When the polynomial $p(x) = x^4 + ax + 2$ is divided by $x^2 + 1$ the remainder is $2x + 3$. Find the value of a.

Question 42

(a) (i) If $z_1 = 2cis\left(\dfrac{\pi}{4}\right)$ find z_1^4.

(ii) Draw z_1 on an Argand diagram.

(b) How many solutions to $z^4 = -16$ are there if $z \in \mathbb{C}$?

(c) Solve $z^5 + 16z = 0$, $z \in \mathbb{C}$.

Question 43

The two square roots of $5 - 12i$ take on the form $x + iy$, $x, y \in \mathbb{R}$. Find both values of z for which $z^2 = 5 - 12i$, giving your answer in the form $x + iy$, $x, y \in \mathbb{R}$.

Question 44

Find the four fourth roots of the complex number $1 + \sqrt{3}i$, giving your answer in the form $re^{i\theta}, -\pi < \theta \leq \pi$.

Question 45

Find a and b, where $a, b \in \mathbb{R}$ in each of the following cases

(a) $a + bi = i$.

(b) $(a + bi)^2 = i$.

(c) $(a + bi)^3 = i$.

Question 46

If $(a + bi)^n = x + iy, n \in \mathbb{Z}^+$, show that $x^2 + y^2 = (a^2 + b^2)^k$ where k is a function of n. Expresss k in terms of n.

FUNCTIONS AND EQUATIONS

Question 1

Consider the function $g(x) = \frac{1}{\sqrt{x-3}}, x \in X$.

(a) Find (i) $g(4)$.
(ii) $g(12)$.
(iii) $g(9)$.

(b) Find the largest set X for which $g(x)$ is defined.

(c) Sketch the graph of $g(x)$ using its maximal domain X.

Question 2

Consider the functions $f(x) = \frac{1}{x-1}$ and $g(x) = x^3$.

(a) Determine the maximal domain of (i) $f(x)$.
(ii) $g(x)$.

(b) (i) Justify the existence of $(g \circ f)(x)$.
(ii) Find $(g \circ f)(x)$.

Question 3

Consider the function $f(x) = \sqrt{7-x}$.

(a) Find (i) $f(3)$.
(ii) $f(7)$.

(b) What is the maximal domain of $f(x)$.

(c) Find (i) $g(x) = \frac{1}{f(x)}$ and determine the implicit domain of $g(x)$.
(ii) $h(x) = f^{-1}(x)$ and determine the implicit domain of $h(x)$.

Question 4

Given the function $f(x) = \dfrac{1}{1+e^{2x}}$, $x \in \mathbb{R}$ and the fact that $f(x) + f(-x) = ax + b$, where $a, b \in \mathbb{R}$, find the values of a and b.

Question 5

Consider the function $f(x) = 3x^2 - 6x + 7$.

(a) Express $f(x)$ in the form $a(x-h)^2 + k$, where $a, b, c \in \mathbb{Z}^+$.

(b) For the graph of $f(x)$ write down the
(i) coordinates of its vertex.
(ii) equation of the axis of symmetry.

(c) (i) Find the coordinates of the y-intercept.
(ii) Sketch the graph of $f(x)$, clearly labelling the vertex, y-intercept and axis of symmetry.

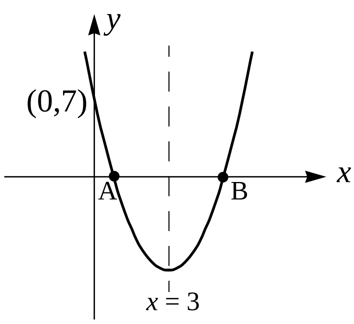
Question 6

(a) Determine the maximal domain of $f(x) = \ln(x+1)$.

(b) If $g(x) = e^x$,
(i) justify the existence of $(f \circ g)(x)$.
(ii) find $(f \circ g)(x)$.
(iii) state the maximal domain of $(f \circ g)(x)$.

Question 7

The graph of the function $g(x) = 2x^2 - 12x + 7$ is shown below. The coordinates of B are $\left(a + \sqrt{\frac{b}{c}}, 0\right)$ where $a, b, c \in \mathbb{Z}$. Find the values of a, b and c.



Question 8

Consider the function $g(x) = x + \frac{1}{x}, x > 0$.

(a) (i) Find $g(2)$.

(ii) Find the x-value for which $g(x) = 2$.

(b) The composite function $(g \circ g)(x) = \frac{f(x)}{x(1+x^2)}, x > 0$. Find $f(x)$.

Question 9

(a) Find the $(f \circ g)(x)$ where $g(x) = e^x + 1, x \in \mathbb{R}$ and $f(x) = (x-1)\ln(x-1), x > 1$.

(b) Find $\{x : (f \circ g)(x) = 2x\}$.

(c) What would the solution to $(f \circ g)(x) = 2x$ be if $g(x) = e^x + 1, x > 0$?

Question 10

(a) Find the inverse function, f^{-1}, of $f(x) = \frac{1}{x-2}, x > 2$.

(b) On the same set of axes, sketch the graphs of $f(x)$ and $f^{-1}(x)$.

(c) Find the coordinates of the point of intersection of $f(x)$ and $f^{-1}(x)$.

Question 11

Consider the function $f_k(x) = \frac{e^{-kx}}{2 + e^{-x}}, x \in \mathbb{R}$.

(a) Find (i) $\{x : f_2(x) = 1\}$.

(ii) $\left\{x : f_1(x) = \frac{1}{2}\right\}$.

(b) For what values of a will $\left\{x : f_1(x) = \frac{1}{a}\right\} = \mathbb{R}$.

Question 12

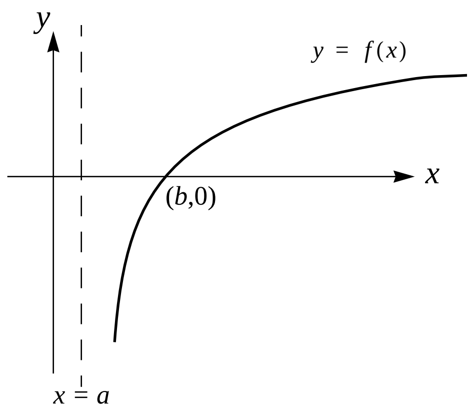
(a) On the same set of axes, sketch the graphs of $f(x) = x^2 - 4$ and $g(x) = \sqrt{x-2}$.

(b) What is the domain of $(f \circ g)(x)$?

(c) Find (i) $(f \circ g)(x)$.
(ii) the range of $(f \circ g)(x)$.

Question 13

The graph of $f(x) = -1 + \ln(x-1), x > a$ is shown.



(a) Find the values of a and b.

(b) Find $f^{-1}(x)$.

(c) Sketch the graph of $f^{-1}(x)$.

Question 14

Find the maximal domain of $h(x) = \dfrac{1}{\ln(\ln x)}$.

Question 15

(a) On the same set of axes sketch the graphs of $f(x) = 5x + 2$ and $g(x) = 3x^2$.

(b) Find (i) $\{x : f(x) = g(x)\}$.
(ii) $\{x : 3x^2 \leq 5x + 2\}$.

Question 16

Consider the function $f : A \rightarrow \mathbb{R}$, where $f(x) = \log_3(3x + 2)$.

(a) Find the largest set A for which f is defined.

(b) (i) Define fully, the inverse, f^{-1}.
(ii) Sketch the graph of $f^{-1}(x)$.
(iii) If $g(x) = 3^x$, express $f^{-1}(x)$ in terms of $g(x)$.

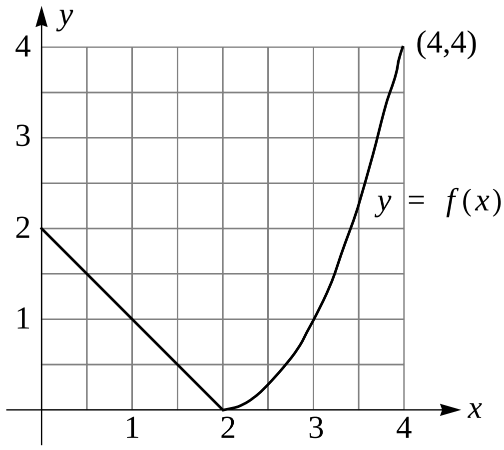
Question 17

For the graph shown below, sketch, on different sets of axes, the graphs of

(a) $y = f(x+1)$.

(b) $\frac{1}{2}y = f(x)$.

(c) $y = f(x) - 2$.



Question 18

Let $f(x) = \frac{1}{x}$, $x \neq 0$. The graph of $g(x)$ is a translation of the graph of $f(x)$ defined by the matrix $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

(a) Find an expression for $g(x)$
  (i) in terms of $f(x)$.
  (ii) in terms of x.

(b) On the same set of axes, sketch the graphs of
  (i) $f(x)$.
  (ii) $g(x)$, for $x > 3$.

Question 19

The function $g(x) = ax - b$ passes through the points with coordinates $(1, 9)$ and $(-3, 1)$.
Find a and b.

Question 20

Let $f(x) = x^2, x \in \mathbb{R}$.

(a) Sketch the graph of f.

The graph of f is transformed to the graph of g by a translation $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

(b) Find an expression for $g(x)$ in terms of
(i) $f(x)$.
(ii) x.

(c) (i) Find $\{x : g(x) = 0\}$.
(ii) Find $g(0)$.
(iii) Sketch the graph of $g(x)$.

Question 21

Given that $f(x) = \dfrac{x-1}{x+1}$, find (a) $f(1)$.

(b) $f^{-1}(-1)$.

Question 22

The function $f(x)$ undergoes a transformation defined by the matrix $\begin{pmatrix} a \\ -b \end{pmatrix}$ to produce the new function $g(x)$.

(a) Express $g(x)$ in terms of $f(x)$.

(b) If $f(x) = x^2 - 2x + 4$ and $g(x) = x^2 - 2x + 8$, find a and b.

Question 23

Let $f(x) = 2e^{-x} + 3, x \in \mathbb{R}$, find (a) $f(-1)$.

(b) $f^{-1}(4)$.

Question 24

Let $f(x) = x^3$ and $g(x) = x - 1$. Find

(a) $(f \circ g)(1)$.

(b) $(g \circ f)(2)$.

Question 25

Consider the function $f(x) = \begin{cases} x - 2, & \text{if } x < 0 \\ (x-1)^2 - 3k, & \text{if } x \geq 0 \end{cases}$.

(a) Find the value of k for which $f(x)$ is continuous for $x \in \mathbb{R}$.

(b) Using the value of k in (a), sketch the graph of $f(x)$, clearly labelling all intercepts with the axes.

Question 26

Find all real values of k so that the graph of the function $h(x) = 2x^2 - kx + k$, $x \in \mathbb{R}$ cuts the x-axis at two distinct points.

Question 27

(a) Sketch the graph of $f(x) = x^2 - 4x + 5$, clearly showing and labelling the coordinates of its vertex.

(b) Find (i) $\lim\limits_{x \to \infty} \dfrac{4}{x^2 - 4x + 5}$

(ii) $\lim\limits_{x \to -\infty} \dfrac{4}{x^2 - 4x + 5}$

(c) Given that $g(x) = \dfrac{4}{f(x)}$,

(i) state the maximal domain of $g(x)$.

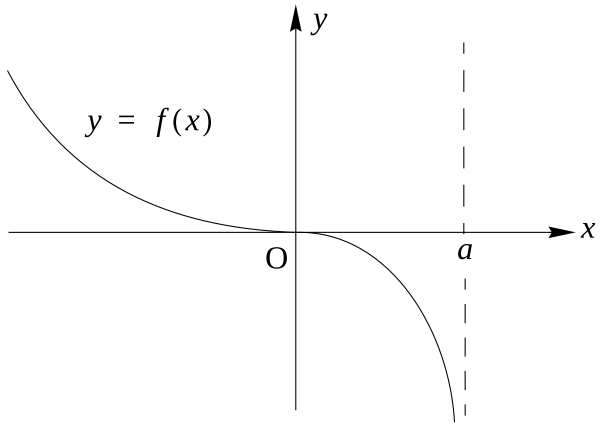
(ii) sketch the graph of $g(x)$.

(iii) find the range of $g(x)$.

Question 28

(a) On the same set of axes sketch the curves of $y = |x|$ and $y = \ln(1-x)$.

The graph of the function $f(x) = |x|\ln(1-x),\ x < a$ is shown below.



(b) Find the value of *a*.

(c) Sketch the graph of (i) $y = |f(x)|$.
(ii) $y = f(|x|)$.
(iii) $y = \dfrac{1}{f(x)}$.

(d) Sketch the graph of $g(x) = |x|\ln(2-x)$.

Question 29

(a) (i) Sketch the graph of $f(x) = |2x+1|$.
(ii) Solve $|2x+1| = 3$.
(iii) Hence solve for x if $g(x) = 0$ where $g(x) = (2x+1)^2 - |2x+1| - 6$.

(b) For what values of x will $(2x+1)^2 - |2x+1| > 6$?

Question 30

(a) (i) $\dfrac{|x-3|}{2x} = \begin{cases} f(x), & x \geq 3 \\ g(x), & x < 3 \end{cases}$. Find $f(x)$ and $g(x)$.

(ii) Sketch the curve of $y = \dfrac{|x-3|}{2x}$.

(b) Find all real values of x that satisfy the inequality $\dfrac{|x-3|}{2x} < 1$.

Question 31

(a) (i) Sketch the curve $y = (x+2)|x+2|$.

(ii) Hence find all real values of x for which $(x+2)|x+2| < 9$.

(b) Find $\{x|\ (x+2)|x-2| < 3\}$.

Question 32

The polynomial $p(x) = ax^3 - x^2 + bx + 6$ is divisible by $(x+2)$ and has a remainder of 10 when divided by $(x+1)$.

(a) Find the values of a and b.

(b) Find $\{x|\ p(x) < 6 - 7x\}$.

Question 33

Consider the curve with equation $y = \dfrac{x}{x^2+4}$.

(a) Show that $yx^2 - x + 4y = 0$.

(b) **Hence**, given that $-a \le y \le a$ for all real values of x, find a.

Question 34

Find the value of a which makes the function defined by $f(x) = \begin{cases} x+7, & x \le 3 \\ a \times 2^x, & x > 3 \end{cases}$ continuous for all real values of x.

Question 35

(a) Sketch the graph of the polynomial $p(x) = x^3 - x^2 - 5x - 3$.

(b) Find all real values of x for which

(i) $x^3 - x^2 - 5x - 3 < 4(x-3)$.

(ii) $x^3 - x^2 - 5x - 3 > (x+1)(x-3)$.

Question 36

Given that $a + b = 2$, find the values of *a* and *b* so that the function $f(x) = \begin{cases} ax + 1, x < \dfrac{\pi}{2} \\ \sin x + b, x \geq \dfrac{\pi}{2} \end{cases}$

is continuous for all values of *x*.

Question 37

Given that $h(x) = 9^x + 9$ and $g(x) = 10 \times 3^x$, find $\{x| \ h(x) < g(x)\}$.

CIRCULAR FUNCTIONS AND TRIGONOMETRY

Question 1

Given that $0 \le \theta \le \frac{\pi}{2}$ and $\tan\theta = \frac{3}{4}$, find

(a) $\cos\theta$.

(b) $\sin 2\theta$.

(c) $\tan\left(\frac{\pi}{2} - \theta\right)$.

Question 2

Given that $0 < \theta \le \frac{\pi}{2}$, arrange, in increasing order, $\sin\theta$, $\frac{1}{\sin\theta}$, $\sin^2\theta$.

Question 3

(a) If $0 < \theta < 90°$ and $\cos\theta = \frac{1}{3}a$, find $\sin\theta$.

(b) Express $\frac{\pi}{5}$ in degrees.

(c) Evaluate $\cos 300° \cos 30°$

(d) Express in terms of $\tan\theta$, $\frac{\sin(\pi - \theta)}{\cos\left(\frac{\pi}{2} + \theta\right)} \cdot \tan\left(\frac{3\pi}{2} - \theta\right)$.

Question 4

(a) Express $\frac{1}{\cos\theta - 1} - \frac{1}{\cos\theta + 1}$ in terms of $\sin\theta$.

(b) Solve $\frac{1}{\cos\theta - 1} - \frac{1}{\cos\theta + 1} = -8$, $0° < \theta < 360°$

Question 5

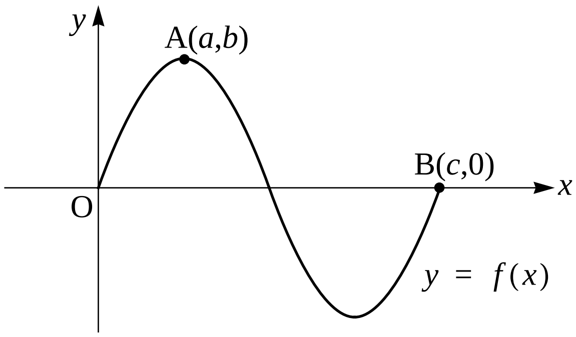
(a) Given that $\cos\theta\sin\theta = \frac{1}{2}$, evaluate $(\cos\theta - \sin\theta)^2$.

(b) Find all values of θ such that (i) $\sin^2\theta - \sin\theta = 0, 0 \le \theta \le 2\pi$.

(ii) $\sin^2\theta - \sin\theta = 2, 0 \le \theta \le 2\pi$.

Question 6

The figure below shows the graph of $f(x) = 2\sin\left(\frac{x}{2}\right)$.

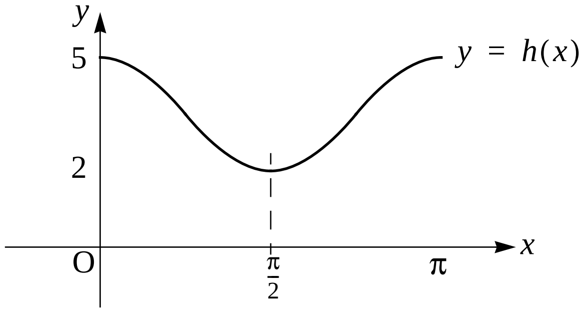


(a) Find a, b and c.

(b) Solve for x, where $f(x) = \sqrt{3}, 0 \le x \le c$.

Question 7

Consider the graph of the function $h(x) = a\cos(bx) + c$:



Find the values a, b and c.

Question 8

(a) State the range of $2\sin\theta$.

(b) Find (i) the smallest value of $\frac{1}{3+4^{\sin\theta}}$.

(ii) the largest value of $\frac{1}{3+4^{\sin\theta}}$.

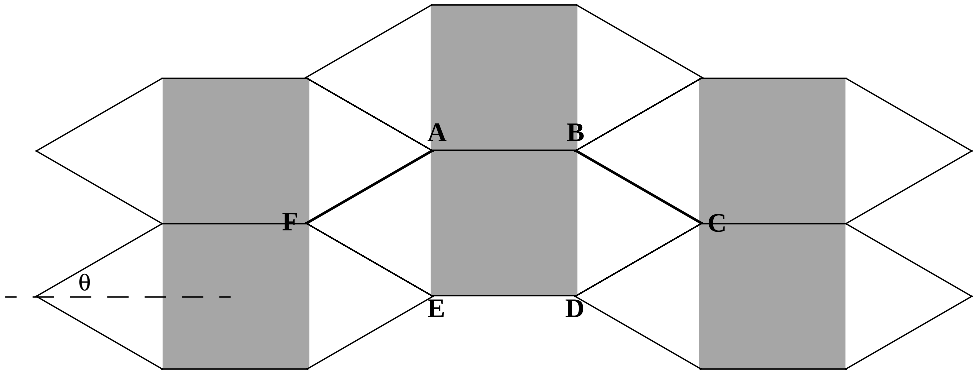
Question 9

(a) Find all values of x such that $2\cos(x) + \sin(2x) = 0, x \in [0, 2\pi]$.

(b) How many solutions to $2\cos(x) + \sin(2x) = \cos x$ are there in the interval $[0, 2\pi]$.

Question 10

Part of a particular tile pattern, made up by joining hexagonal shaped tiles, is shown below.



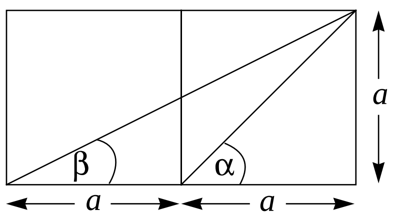
The side lengths of every hexagon is x cm.

(a) Find in terms of x and θ the length of [AE].

(b) If the area of the shaded region in any one of the hexagons is 9 cm^2 and $\sin\theta = \frac{1}{8}$, find the value of x.

Question 11

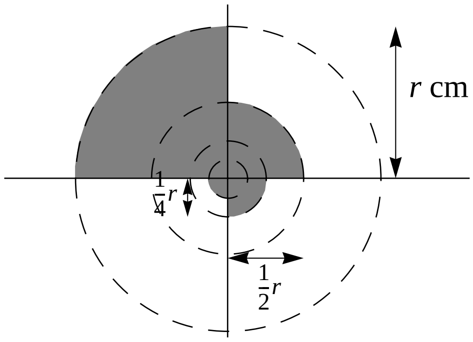
For the diagram shown alongside, the value of $\sin(\alpha - \beta) = \frac{1}{\sqrt{k}}$, where $k \in \mathbb{Z}^+$.



Find the value of k.

Question 12

The following diagram shows continually decreasing quarter circles, where each successive quarter circle has a radius half that of the previous one.



Let A_1 equal the area of the quarter circle of radius r, A_2 equal the area of the quarter circle of radius $\frac{1}{2}r$, A_3 equal the area of the quarter circle of radius $\frac{1}{4}r$ and so on.

(a) Find (i) A_1 in terms of r.

(ii) A_2 in terms of r.

(iii) A_3 in terms of r.

(b) If $A_1 + A_2 + A_3 = \frac{525}{64}\pi$, find r.

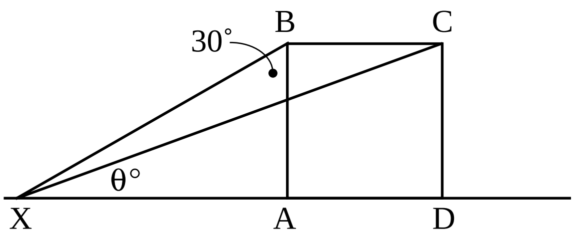
(c) If quarter circles are drawn and shaded in indefinitely, find r if $\sum_{i=1}^{\infty} A_i = 27\pi$.

Question 13

A pole of length b metres resting against a wall makes an angle of 60˚ with the ground. The end of the pole making contact with the ground starts to slip away from the wall until it comes to rest, 1 m from its initial position, where it now makes an angle of 30˚ with the ground. Find the value of b.

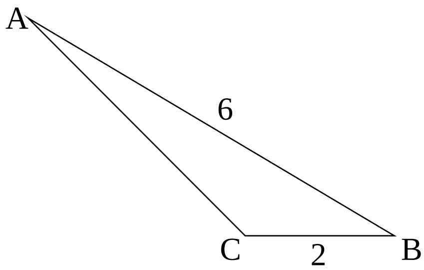
Question 14

The figure below shows a square ABCD. If $\angle XBA = 30°$ and $\angle CXA = \theta°$, find $\tan\theta°$.



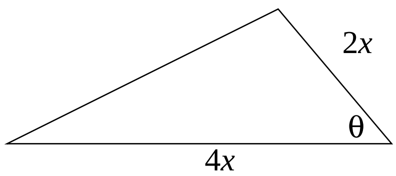
Question 15

In the diagram below, $\cos B = \frac{3}{8}$, AB = 6 and BC = 2, $AC = \sqrt{b}$, find b.



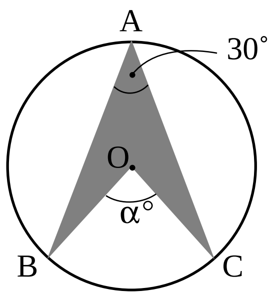
Question 16

The area of the triangle shown is 27 sq. units. Given that $\sin\theta = \frac{3}{4}$, find x.



Question 17

A circular badge of radius 2 cm has the following design:



(a) State the value of α.

(b) Find (i) the length of the minor BC.
(ii) the area of the minor sector OBC.

(c) Find the area of the shaded region shown.

Question 18

If $0° \le x \le 90°$, solve each of the following

(a) $\cos x = \sin 36°$.

(b) $\cos x = \sin x$.

(c) $\cos 2x = \sin x$.

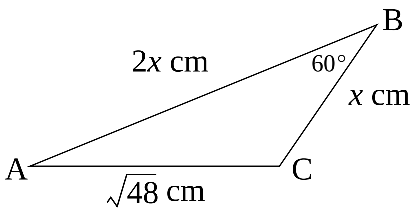
Question 19

(a) If $\tan\theta = a$, express $\dfrac{4\sin\theta}{5\cos\theta - \sin\theta}$ in terms of a.

(b) **Hence** find the value of θ if $\dfrac{4\sin\theta}{5\cos\theta - \sin\theta} = 1$, $0 < \theta < \dfrac{\pi}{2}$.

Question 20

Consider the triangle ABC:

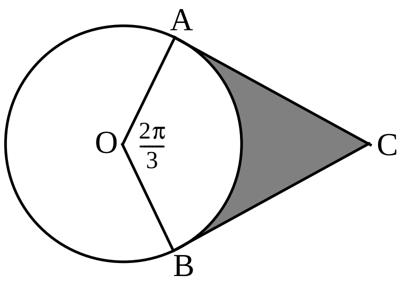


(a) Find x.

(b) Find the area of ΔABC

Question 21

The segments [CA] and [CB] are tangents to the circle at the points A and B respectively. If the circle has a radius of 4 cm and $\angle AOB = \dfrac{2\pi}{3}$.

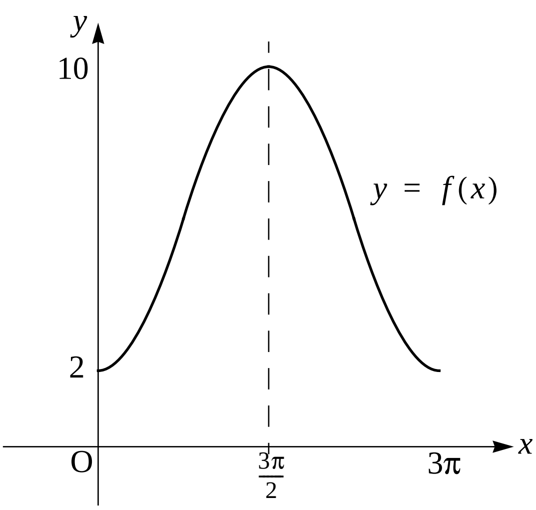


(a) Find the length of [OC].

(b) Find the area of the shaded region.

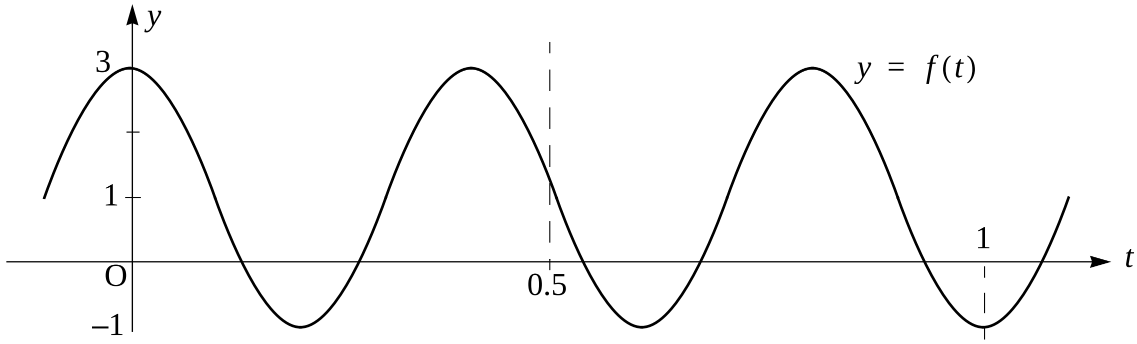
Question 22

The graph of $f(x) = a\cos(bx) + c$, $0 \le x \le \pi$ is shown below. Find the values a, b and c.



Question 23

(a) Part of the graph of the function defined by $y = f(t)$ is shown below.



(i) What is the period of $f(t)$?

(ii) Given that $f(t) = a\cos(kt) + c$, find a, k and c.

(b) What is the least positive value of x for which $\cos(2x) = \sqrt{3}\sin(2x)$?

(c) Let $g(x) = m\cos(x) + n$, $n, m \in \mathbb{R}$ and $m > 0$. Write an expression for n in terms of m if $g(x) < 0$ for all real values of x.

(d) Sketch the graph of $h :]-\pi,\pi[\mapsto \mathbb{R}$, where $h(x) = \tan\left(\frac{x}{2}\right) + 1$, clearly determining and labelling the x-intercepts.

(e) Find the sum of the solutions to $\cos\left(\frac{x}{2}\right) = \frac{1}{2}\sqrt{3}$, $x \in [0, 4\pi]$.

Question 24

Destructive interference generated by two out of tune violins results in the production of a sound intensity, $I(t)$, given by the equation

$$I(t) = 30 + 4\cos\left(\frac{4\pi}{3}t\right) \text{ dB (decibels)}$$

where t is the time in seconds after the violins begin to sound.

(a) What is the intensity after 1 second?

(b) What is the least intensity generated?

(c) When does the intensity first reach 32 dB?

(d) What time difference exists between successive measures of maximum intensity?

Question 25

The rabbit population, $N(t)$, over a ten year cycle in a small region of South Australia fluctuates according to the equation

$$N(t) = 950\cos(36t°) + 3000, \ 0 \le t \le 10$$

where t is measured in years.

(a) Find the rabbit population after 2 and a half years.

(b) What is the minimum number of rabbits in this region that is predicted by this model?

(c) Sketch the graph of $y = N(t)$, $0 \le t \le 10$.

(d) For how long, over a 10 year cycle, will the rabbit population number at most 3475?

Question 26

Solve the equation for x, where $\sin 2x = \sqrt{3}\cos x$, $0 \le x \le \pi$.

Question 27

(a) Expand $\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$.

(b) Solve $\cos^2\theta + \sin\theta\cos\theta = -(\cos\theta + \sin\theta)$, $0 \le \theta \le \pi$.

Question 28

If $\sin A = \frac{\sqrt{2}}{4}$ and $\cos B = \frac{\sqrt{3}}{4}$ where both A and B are acute angles, find

(a) (i) sin2A.
    (ii) cos2B.

(c) sin(A + B).

Question 29

(a) (i) If $T(x) = 6 - (x-1)^2$, what is the maximum value of $T(x)$?
    (ii) Express $\sin^2 x + 2\cos x + 6$ in the form $a + b(\cos x - 1)^2$.

(b) What is the maximum value of $\sin^2 x + 2\cos x + 6$?

Question 30

(a) Let $s = \sin\theta$. Show that the equation $15\sin\theta + \cos^2\theta = 8 + \sin^2\theta$ can be expressed in the form $2s^2 - 15s + 7 = 0$.

(b) Hence solve $15\sin\theta + \cos^2\theta = 8 + \sin^2\theta$ for $\theta \in [0, 2\pi]$.

Question 31

Find $\{x |\ 4\sin^3 x + 4\sin^2 x - \sin x - 1 = 0, 0 \le x \le 2\pi\}$

Question 32

If $\sin A = 2\sin(\theta - A)$, find the values of a and b such that $\tan A = \dfrac{a\sin\theta}{1 + b\cos\theta}$

Question 33

(a) Show that (i) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(ii) $\sin\theta° = \cos(90° - \theta°)$.

(b) Noting that $2 \times 18 = 36$ and $3 \times 18 = 54 = 90 - 36$, find the exact value of $\sin 18°$.

Question 34

Given that $u = \dfrac{1 + \sin\theta}{\cos\theta}$, prove that

(a) $\sin\theta = \dfrac{u^2 - 1}{u^2 + 1}$.

(b) $\cos\theta = \dfrac{2u}{u^2 + 1}$

Question 35

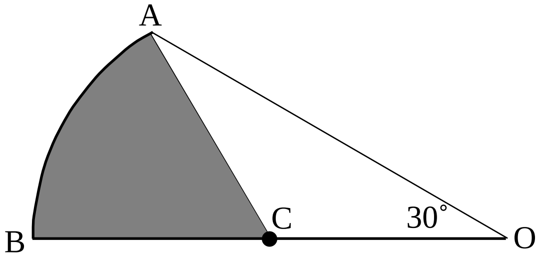
Given that $\sin x \sin y = \dfrac{\sqrt{3}}{4}$ and $\cos x \cos y = \dfrac{\sqrt{3}}{4}$, find all real x and y that satisfy these equations simultaneously, where $0 \le x \le \pi$ and $0 \le y \le \pi$.

Question 36

Given that $a\sec\theta = 1 + \tan\theta$ and $b\sec\theta = 1 - \tan\theta$, show that $a^2 + b^2 = k$ and hence state the value of k.

Question 37

A sector OAB of radius 4 cm is shown below, where the point C is the midpoint of [OB].



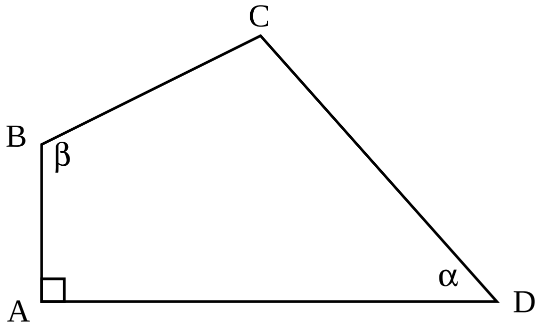
(a) Find the area of the shaded region.

(b) Where, along [OB], should the point C be placed so that [AC] divides the area of the sector OAB in two equal parts.

Question 38

In the figure shown, DC = a cm, BC = b cm, $\angle ABC = \beta$ and $\angle ADC = \alpha$.

If $AD = x\cos\alpha + y\sin\beta$, find x and y in terms of a and b.



Question 39

Point A is x metres due south of a vertical pole OP and is such that the angle of elevation of P from A is 60˚. Point B is due east of the pole OP and makes an angle of elevation to P of 30˚. The points O, A and B all lie on the same horizontal plane. A metal wire is attached from A to P and another wire is attached from B to P.

(a) Find an expression in terms of x for
 (i) AP.
 (ii) BP.
 (iii) AB.

(b) The sine of the angle between the two wires is $\frac{a\sqrt{b}}{c}$ where $a, b, c \in \mathbb{Z}^+$. Find a, b and c.

Question 40

Consider the function $f(x) = 2\sin\left(x - \frac{\pi}{2}\right) + 1,\ 0 \le x \le \pi$.

(a) (i) Find the range of *f*.
  (ii) Sketch the graph of *f*.

(b) Find f^{-1}.

(c) Sketch the graph of f^{-1}.

Question 41

(a) (i) Sketch on the same set of axes the graphs of $f(x) = \text{Arccos}(x)$ and $g(x) = \text{Arcsin}(x)$.
  (ii) Hence sketch the graph of $h(x) = f(x) + g(x)$.
  (iii) Deduce the value of $\text{Arccos}(x) + \text{Arcsin}(x)$ for $-1 \le x \le 1$.

(b) Find $\sin\left(\text{Arccos}\left(\frac{4}{5}\right) + \text{Arctan}\left(-\frac{4}{5}\right)\right)$

MATRICES

Question 1

Let the matrix $A = \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix}$ define a 2×2 matrix.

(a) Find (i) $|A^2|$.

(ii) A^{-1}.

(b) If $B = \begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$, find the matrix X given that $2X + B = A$.

Question 2

Given that $A = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$

(a) find (i) AB.

(ii) $A^2B - A$.

(b) If $xA - yB = \begin{bmatrix} 5 & 6 \\ 9 & 0 \end{bmatrix}$ where $x, y \in \mathbb{R}$, find x and y.

Question 3

Consider the matrices $A = \begin{bmatrix} 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 0 \\ 2 & 4 \end{bmatrix}$. Find, where possible,

(a) AB.

(b) $A + B$.

(c) BC.

(d) CB.

(e) CA.

Question 4

Find x given that (i) $\begin{bmatrix} x^2 & 7 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 7 \\ 4 & x+3 \end{bmatrix}$.

(ii) $\begin{bmatrix} x^2+6 & 7 \\ x+2 & 3 \end{bmatrix} = \begin{bmatrix} 5x & 7 \\ 4 & 3 \end{bmatrix}$.

Question 5

(a) Find $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 5 & -4 \\ 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix}$.

(b) (i) Set up the system of simultaneous equations $\begin{aligned} 2x + z &= 5 \\ 2x + y - z &= 4 \\ 3x + y - z &= 3 \end{aligned}$ in the form $AX = B$,

where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(ii) **Hence** solve for x, y and z.

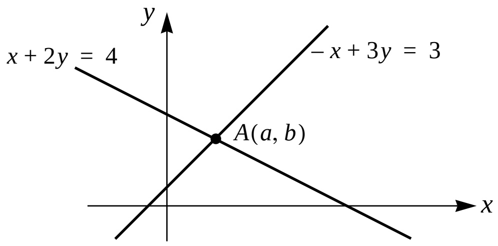
Question 6

If $\begin{bmatrix} 2x+8 & 4 \\ 0.5x & y \end{bmatrix} = \begin{bmatrix} 3x-y & 4 \\ 5 & y \end{bmatrix}$, find x and y.

Question 7

(a) Given that $A = \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}$, find A^{-1}.

(b) **Hence** find the coordinates (a, b) of point A in the diagram shown below.



Question 8

If $A = \begin{bmatrix} \cos^2\theta & -\cos\theta \\ \sin\theta & \sin^2\theta \end{bmatrix}$ and $B = \begin{bmatrix} \sin^2\theta & \cos\theta \\ -\sin\theta & \cos^2\theta \end{bmatrix}$, find

(a) (i) $A + B$.
(ii) $(A+B)^{-1}$.
(iii) $A - B$.

(b) For what values of θ where $0 < \theta < \pi$, does $A - B = \begin{bmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix}$.

Question 9

Consider the function $T(x) = 3x^2 + 4x + I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(a) If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, find $T(A)$.

(b) (i) Find $(3A + I)(A + I)$.
(ii) **Hence** find $3A^2 - 4A + I$.

Question 10

Given that $A = \begin{bmatrix} x & 1 \\ 2 & x-1 \end{bmatrix}$, for what values of x will A be singular?

Question 11

(a) Given that $A = \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$, find

(i) $|A|$.
(ii) A^{-1}.

(b) **Hence** solve the system of simultaneous equations $\begin{aligned} 7x + 4y &= 4 \\ 3x + 2y &= -2 \end{aligned}$.

Question 12

If $A = \begin{bmatrix} 0 & 3 \\ -1 & 1 \end{bmatrix}$, find scalars α, β and γ, where $\alpha \neq 0, \beta \neq 0, \gamma \neq 0$, for which

$$\alpha A^2 + \beta A + \gamma I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Question 13

(a) (i) Let $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, find AB.

(ii) **Hence** find A^{-1}.

(b) **Hence** solve the system of simultaneous equations
$$\begin{aligned} -x + y + z &= 3 \\ x - y + z &= 4 \\ x + y - z &= 5 \end{aligned}.$$

Question 14

Find x if $|A| = 0$ and $A = \begin{bmatrix} x-1 & 1 & 2 \\ 2 & x-3 & 1 \\ -1 & 1 & x-2 \end{bmatrix}$.

Question 15

(a) Find, in terms of a, $\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 5 & 2 & 3 \\ 1 & 1 & (a-2) \end{vmatrix}$.

(b) Consider the system of equations
$$\begin{aligned} 3x + y + 2z &= 4 \\ 5x + 2y + 3z &= 6 \\ x + y + (a-2)z &= 0 \end{aligned}.$$

(i) Solve the system of equations when $a = 3$.

(ii) For what values(s) of a will the system of equations have exactly one solution.

Question 16

Consider the system of equations $\begin{aligned} 3x + 2y - z &= 1 \\ x + y + z &= 2 \\ kx + 2y - z &= 1 \end{aligned}$.

(a) Find the value of k for which the system of equations has more than one solution.

(b) Solve the system of equations for the value of k in part (a).

Question 17

(a) If $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 3 & -13 & p \\ 0 & p+3 & -1 \end{vmatrix}$, find

(i) an expression for Δ in terms of p.

(ii) the value(s) of p for which $\Delta = 0$.

(b) For what values of p will the system of equations $\begin{aligned} x + y + z &= 2 \\ 3x - 13y + pz &= p + 1 \\ (p+3)y - z &= 0 \end{aligned}$ have a unique solution?

(c) For each value of p for which a unique solution does not exist, determine whether a solution exists and if it does, find all solutions.

Question 18

Consider the matrices $A = \begin{bmatrix} 1 & a & 2 \\ a & 1 & 1 \\ 2 & -2 & a+2 \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 1 \\ 8 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $a \in \mathbb{R}$.

(a) Show that the equation $AX = B$ has a unique solution when a is neither 0 nor –1.

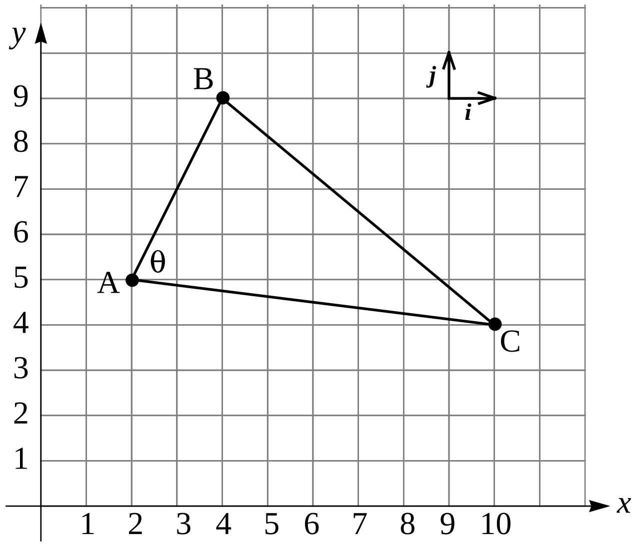
(b) (i) If $a = 0$, find the particular solution for which $x + y + z = 0$.

(ii) If $a = -1$, show that the system of equations has no solution.

VECTORS

Question 1

The points A, B and C are shown in the diagram below



(a) Find in terms of the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ (i) $\overrightarrow{AB}$.

(ii) $\overrightarrow{AC}$.

(b) (i) Find $\overrightarrow{AB} \bullet \overrightarrow{AC}$.

(ii) Hence find $\cos^2\theta$.

Question 2

ABCD is a quadrilateral where P, Q and R are the midpoints of [AB], [BC] and [CD] respectively. If $\overrightarrow{AD} + \overrightarrow{BC} = k\overrightarrow{PR}$, where k is a positive integer, find k.

Question 3

Given that $\boldsymbol{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, find

(a) $\boldsymbol{a} + 2\boldsymbol{b} + \boldsymbol{c}$.

(b) $\boldsymbol{a} \bullet \boldsymbol{b}$.

(c) $\boldsymbol{a} - \frac{1}{3}(\boldsymbol{c} - \boldsymbol{b})$.

(d) $|\boldsymbol{c}|$.

Question 4

Consider the vector $\boldsymbol{r} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$.

(a) Find (i) $\boldsymbol{r} \bullet \boldsymbol{r}$.
(ii) $\hat{\boldsymbol{r}}$.

(b) If $\boldsymbol{a} = \begin{pmatrix} x \\ 1 \\ y \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, find x and y given that $\boldsymbol{r} = -4\boldsymbol{a} + 2\boldsymbol{b}$.

Question 5

If $\boldsymbol{a} = 2\boldsymbol{i} - 6\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$, find

(a) (i) $|\boldsymbol{a}|$ (ii) $|\boldsymbol{b}|$.

(b) $|\boldsymbol{b} - \boldsymbol{a}|$.

(c) $(\boldsymbol{a} - \boldsymbol{b}) \bullet (\boldsymbol{a} + \boldsymbol{b})$.

Question 6

The vector $\boldsymbol{a} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$ is a linear combination of the vectors $\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\boldsymbol{y} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\boldsymbol{z} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. That is, $\boldsymbol{a} = \alpha\boldsymbol{x} + \beta\boldsymbol{y} + \gamma\boldsymbol{z}$. Find α, β and γ.

Question 7

The vector $\boldsymbol{a} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0.5 \end{pmatrix}$ makes an angle of 60° with the vector $\boldsymbol{b} = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0.5 \end{pmatrix}$. Find the value of $\sin 2\theta$.

Question 8

If vectors $\boldsymbol{a} = \begin{pmatrix} x \\ 3 \\ -1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ x^2 \\ 1 \end{pmatrix}$ are perpendicular, find x.

Question 9

Given that $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 3$ and that $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$, find

(a) $(\boldsymbol{a} - \boldsymbol{b}) \bullet (\boldsymbol{a} + \boldsymbol{b})$.

(b) $(\boldsymbol{a} + \boldsymbol{b}) \bullet (\boldsymbol{a} + \boldsymbol{b})$.

(c) $(\boldsymbol{a} + \boldsymbol{b}) \bullet (\boldsymbol{a} + \boldsymbol{b}) - (\boldsymbol{a} - \boldsymbol{b}) \bullet (\boldsymbol{a} - \boldsymbol{b})$.

Question 10

(a) The straight line, l_1, passes through the points (2, 3, –1) and (5, 6, 3). Write down the vector equation, $\boldsymbol{r}_1$, of this straight line.

(b) A second line is defined by the vector equation $\boldsymbol{r}_2 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, where μ is a real number. If the angle between the two lines is θ, find the value of $\cos\theta$.

Question 11

The vector equation of line l_1 is given by $\boldsymbol{r}_1 = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$.

(a) If the point P on $\boldsymbol{r}_1$ corresponds to when $t = 1$, find
  (i) the position vector of P.
  (ii) how far P is from the origin O.

A second line, l_2, is defined by the vector equation $\boldsymbol{r}_2 = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k} + s(2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k})$, $s \in \mathbb{R}$.

(b) If l_2 intersects l_1 when $t = 1$, find the value of s.

Question 12

Consider the points $A(0, -1, 5)$, $B(1, 3, 3)$, $C(5, 4, 0)$ and $D(3, 0, 4)$.

(a) Find the vectors (i) $\overrightarrow{AB}$.

(ii) $\overrightarrow{BC}$.

(iii) $\overrightarrow{DC}$.

(b) Find the cosine of the angle between vector $\overrightarrow{AB}$ and vector $\overrightarrow{DC}$.

(c) Calculate $\overrightarrow{AB} \times \overrightarrow{BC}$.

(d) (i) Find the Cartesian equation of the plane Π_1 containing the points A, B and C.

(ii) Find a unit vector which is perpendicular to the plane Π_1.

(e) A second plane, Π_2, contains the points A, B and D. Find the Cartesian equation of the line where Π_2 intersects Π_1.

Question 13

The angle between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is 120° where $|\boldsymbol{x}| = 3$ and $|\boldsymbol{y}| = 2$. If $\boldsymbol{u} = \boldsymbol{x} - 2\boldsymbol{y}$ and $\boldsymbol{v} = 2\boldsymbol{x} + \boldsymbol{y}$, find

(a) $\boldsymbol{u} \bullet \boldsymbol{v}$.

(b) the cosine of the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

Question 14

The two planes $3x - 2y - z - 3 = 0$ and $2x - y - 2z - 5 = 0$ intersect along the line L.

Find the

(a) point on L where $z = 0$.

(b) vector equation of L.

(c) Cartesian equation of L.

Question 15

(a) Find a unit vector perpendicular to both $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{b} = 3\boldsymbol{i} - 6\boldsymbol{j} + 2\boldsymbol{k}$.

(b) Find the sine of the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

Question 16

If the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are coplanar, where $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{c} = \alpha\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$, $\alpha \in \mathbb{R}$, find α.

Question 17

(a) Two vectors, $\boldsymbol{a} = 6\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + \alpha\boldsymbol{j} + \beta\boldsymbol{k}$ where $\alpha, \beta \in \mathbb{R}$ are parallel. Find the values of α and β.

(b) Find the area of the triangle formed by the points A(–3, 2, 4), B(1, –1, 2) and C(2, 1, 5).

Question 18

Find the equation of the plane which passes through the point A(3, 4, –1) and which is normal to the vector $\boldsymbol{n} = -2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$.

Question 19

Consider the planes $\begin{array}{l} \Pi_1 : \boldsymbol{r} \bullet \boldsymbol{n}_1 = 4 \\ \Pi_2 : \boldsymbol{r} \bullet \boldsymbol{n}_2 = 3 \end{array}$, where $\boldsymbol{n}_1 = 2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{n}_2 = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$.

Find the equation of the plane that is perpendicular to both Π_1 and Π_2 and that also passes through the point P(2, 1, –2). Give your **answer in normal vector form**.

Question 20

Two vectors, $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are perpendicular to the planes $\Pi_1 : -x + 2y + z + 3 = 0$ and $\Pi_2 : 3x + 4y + 2 = 0$ respectively.

(a) Find two vectors $\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{n}}_2$.

(b) **Hence** find the cosine of the acute angle between the planes Π_1 and Π_2.

Question 21

(a) Find $\begin{vmatrix} -1 & -2 & 1 \\ -3 & -1 & 4 \\ 1 & 1 & 6 \end{vmatrix}$.

(b) The points A(1, 2, –1), B(–2, 1, 3) and C(2, 3, 5) lie on the plane Π_1. Find the vectors

(i) $\overrightarrow{AB}$.

(ii) $\overrightarrow{AC}$.

(c) Hence find the equation of Π_1.

Question 22

The plane $2x + 3y - z + 4 = 0$ and the line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-1}{-2}$ intersects at the point A. Find the coordinates of A.

Question 23

The line L passing through the points A(2, 5, 4) and B(4, 6, 2) intersects the plane $2x - 3y - 4z = 12$. The inclination of L to this plane is $\theta = \arcsin\left(\frac{a}{\sqrt{b}}\right)$, find a and b, where a and b are smallest positive integers possible.

Question 24

(a) Find the vector equation of the plane containing the vectors $\begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$ and which also includes the point (3, 4, 2).

(b) Find the Cartesian equation of the plane in (a).

STATISTICS AND PROBABILITY

Question 1

A random variable X is distributed normally with a mean of 50 and a variance of 16.

(a) Find $P(X > 50)$.

(b) Find (i) the value of X that is 1.65σ above the mean.
(ii) the standardised normal value corresponding to $X = 64$.

Question 2

A random variable X is distributed normally with a mean of 82 and a standard deviation of 6. Given that $P(Z < 1.6) = 0.945$, correct to 3 decimal places, find $P(X > 91.6)$.

Question 3

The lengths of a particular type of lizard is found to be normally distributed with a mean length of 15 cm and a standard deviation of 2 cm. If the random variable X denotes the lengths of these lizards and $P(Z > 1) = a$, where Z is the normal standard random variable, find

(a) the probability of a lizard being greater than 17 cm long.

(b) $P(|X| < 17)$.

Question 4

The random variable X is normally distributed with a mean of 82 and a standard deviation of 4. Let Z be the standard normal random variable. Use the result that $P(Z < 1.5) = 0.93$, correct to two decimal places, to find

(a) $P(X > 88)$ (to 2 decimal places).

(b) $P(76 < X < 88)$ (to 2 decimal places).

(c) $P(X > 76 | X < 88)$.

Question 5

The length, in minutes, of telephone calls at a small office was recorded over a one month period. The results are shown in the table below.

Length of call (minutes)	Number of calls
$0 < t \le 2$	40
$2 < t \le 4$	60
$4 < t \le 6$	40
$6 < t \le 8$	30
$8 < t \le 10$	20
$10 < t \le 12$	10
Total	**200**

(a) Construct a cumulative frequency graph.

(b) Find (i) the mean length of calls.
(ii) the mode of the length of calls.
(iii) the median.

Question 6

For the data set; $x-4,\ x+4,\ x-6,\ x-6,\ x+6,\ x+12$, find

(a) the median.

(b) the mean.

(c) the variance, σ^2.

Question 7

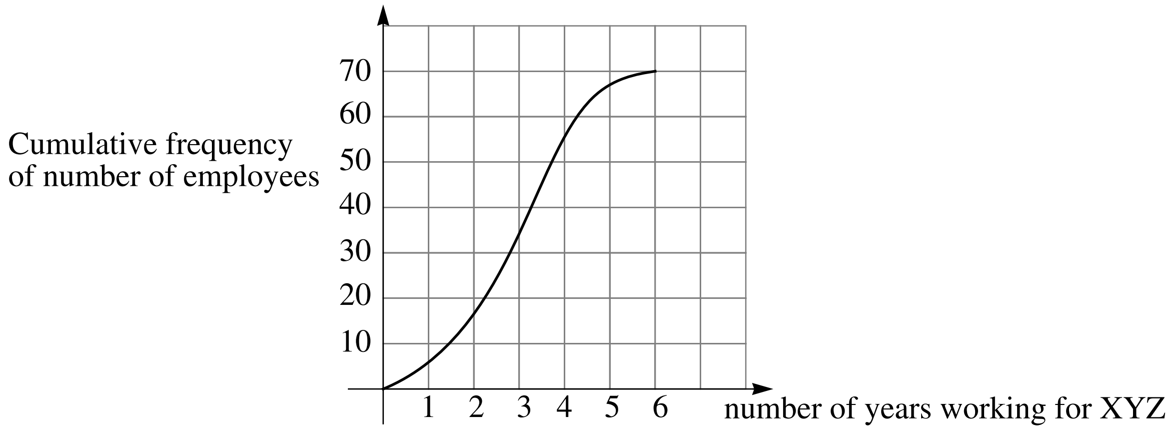
A carton contains 12 eggs of which 3 are known to be bad. If 2 eggs are randomly selected, what is the probability that

(a) one is bad?

(b) both are bad?

Question 8

The following cumulative frequency graph shows the number of years that employees remain at the XYZ company.



(a) Find (i) the median.
(ii) the upper quartile.
(iii) interquartile range.

(b) The probability that an employee serves no more than 4 years at the XYZ company.

Question 9

A regular tetrahedron with its faces numbered 1 to 4 has the following probability distribution.

Score (X)	1	2	3	4
Probability	$\frac{3}{20}$	$\frac{1}{10}$	x	$\frac{1}{5}$

Where the random variable X denotes the number that the tetrahedron lands on, i.e., the score.

(a) Find the value of x.

(b) Find (i) $\mathrm{E}(X)$.
(ii) $\mathrm{Var}(X)$.

(c) The tetrahedron is rolled twice, what is the probability that **the sum** of the scores is
(i) 2.
(ii) 4.

Question 10

The speed of Gab's tennis serves are normally distributed with a mean speed of $102\ \text{km h}^{-1}$. The probability that Gab's serve reaches a speed in excess of $114\ \text{km h}^{-1}$ is 0.08.

(a) Find the probability that Gab's next serve is less than $114\ \text{km h}^{-1}$.

(b) Gab records a serve reaching a speed of $a\ \text{km h}^{-1}$. If there is a chance of 0.08 that Gab's serve reaches a speed of under $a\ \text{km h}^{-1}$, find the value of a.

(c) During a training session Gab practices her serves 150 times. On how many occasions will you expect Gab's serve reach a speed of between $102\ \text{km h}^{-1}$ and $114\ \text{km h}^{-1}$?

Question 11

A game is such that there is a $\frac{2}{5}$ chance of winning \$10.00. To play the game a player must pay \$5.00. Joe decides to play three games in a row.

Find the probability that after his three games Joe

(a) wins \$30.00.

(b) makes a profit of \$5.00.

Question 12

Two boxes containing different coloured balls are such that the probability of obtaining a yellow ball from box A is x and obtaining a yellow ball from box B is $\frac{1}{4}$. A box is first selected at random and then one ball is selected from that box.

(a) A box is first selected at random and then one ball is selected from that box. Find, in terms of x, the probability of obtaining a yellow ball.

(b) After selecting a ball it is placed back into the box it came from and the process, as described in (a), is repeated for a second time.

 (i) Find the probability, in terms of x, that if two yellow balls are observed they both came from box A.

 (ii) Show that if the probability in part (i) is $\frac{2}{3}$ then $8x^2 - 8x - 1 = 0$.

Question 13

Students at Leegong Grammar School are enrolled in either physics or mathematics or both. The probability that a student is enrolled in physics given that they are enrolled in mathematics is $\frac{1}{3}$ while the probability that a student is enrolled in mathematics given that they are enrolled in physics is $\frac{1}{4}$. The probability that a student is enrolled in both mathematics and physics is x.

(a) Find, in terms of x, the probability that a student is enrolled in
  (i) mathematics.
  (ii) physics.

(b) Find the probability that a student selected at random is enrolled in mathematics only.

(c) If three such students are randomly selected, what is the probability that at most one of them is enrolled is mathematics only?

Question 14

The random variable X has the following binomial distribution, X~B(4,p). If $P(X = 3) = \frac{4}{3}p^3$, $p > 0$, find

(a) $P(X = 2)$.

(b) (i) $E(X)$.
  (ii) standard deviation of $\sqrt{2}X$.

Question 15

How many permutations are there of the word RETARD if

(a) they start with RE.

(b) they do not start with RE.

Question 16

Show that $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$.

Question 17

The probabilities that persons A, B and C select the correct answer to the same question are 0.3, 0.5 and 0.4 respectively. Find the probability that

(a) they all answer correctly.

(b) none of A, B and C answer correctly.

(c) only one person answers correctly.

(d) at least one answers correctly.

(e) given that only one answered correctly, it was A.

Question 18

A fair coin is tossed five times.

(a) Find the probability of observing
  (i) exactly one tail.
  (ii) at least four tails.
  (iii) at least one tail.

(b) Find the expected number of tails observed.

Question 19

If the random variable X has a binomial distribution with mean 8, variance $\frac{8}{3}$ and is such that $P(X = 1) = \frac{a}{3^b}$, find the smallest values of a and b where $a, b \in \mathbb{Z}^+$.

Question 20

The average number of people arriving at a funpark is λ per minute. Assuming that the number of people, X, arriving at the funpark is a random variable that follows a Poisson distribution and is such that $P(X = 2) = 3P(X = 1)$, find a and b if $P(X = 2 | X \geq 1) = \frac{a}{e^b - 1}$.

Question 21

If $X \sim \text{Po}(\lambda)$, show that $P(X = r + 1) = \frac{\lambda}{r + 1}P(X = r), r = 0, 1, 2 \ldots$.

Question 22

A radioactive source emits particles at an average rate of one every 10 minutes. The probability that in one hour there are at most 2 particles emitted is ae^{-b}, where a and b are positive integers. Find a and b.

Question 23

Noriko tosses n identical fair coins, where $n > 2$. Find the probability that all of the coins or all but one of them will fall with the same face up.

Question 24

If $P(A) = \frac{3}{4}$, $P(B) = \frac{1}{2}$ and $P(A \cup B) = \frac{7}{8}$, find

(a) $P(A \cap B)$.

(b) $P(A' \cap B)$.

(c) $P(A'|B')$.

Question 25

The probability density function of the continuous random variable X is given by

$$f(x) = \begin{cases} k\sin x, & 0 \le x \le \pi \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find the value of k.

(b) Sketch the probability density function, $f(x)$.

(c) Find (i) $P\left(0 < X < \frac{\pi}{4}\right)$.

(ii) $P\left(X > \frac{2\pi}{3}\right)$.

(d) Find E(X).

(e) Find the median of X.

Question 26

The probability density function of the random variable X is given by

$$f(x) = \begin{cases} \dfrac{k}{1+x^2}, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find the value of the constant k.

(b) Find $P\left(0 < X < \frac{1}{\sqrt{3}}\right)$.

(c) Given that $E(X) = \frac{1}{\pi}\ln(b)$, find the value of b.

(d) (i) Find the cumulative distribution function, $F(x)$.
(ii) Sketch the graph of $F(x)$.
(iii) The median of X is $\tan\left(\frac{\pi}{c}\right)$, find the value of c.

Question 27

The probability density function of the random variable X is given by

$$f(x) = \begin{cases} \dfrac{1}{(1+x)^2}, & x \ge 0 \\ 0, & \text{otherwise} \end{cases}.$$

(a) (i) Find the cdf, $F(x)$.
(ii) Sketch the graph of $F(x)$.

(b) Find an expression for the q-quantile, c_q.

(c) Hence, find (i) the median.
(ii) the interquartile range.
(iii) the 95th percentile.

Question 28

Given that $X \sim \mathrm{B}\left(3, \frac{2}{3}\right)$ and $Y \sim \mathrm{B}\left(4, \frac{1}{3}\right)$, find $P(X + Y = 2)$.

CALCULUS

Question 1

Find the equation of the tangent to the curve $y = 2\ln(x+1)$ at the point where $x = 0$.

Question 2

(a) Find $f'(1)$ for (i) $f(x) = \sqrt{x^2+3}$.

(ii) $f(x) = e^{\ln(x^2)}$.

(b) Find $f''(1)$ if $f(x) = x(x+1)^{\frac{3}{2}}$.

Question 3

Use first principles to find the derivative of $f(x) = x^3 + 1$.

Question 4

Let $y = x\tan(x)$.

(a) Evaluate $\frac{dy}{dx}$ when $x = \frac{\pi}{6}$.

(b) Find the equation of the tangent to $y = x\tan(x)$ when $x = \frac{\pi}{4}$.

Question 5

The tangent to the curve $y = x^2 + 2$ has equation $y = -4x + a$. Find the value of a.

Question 6

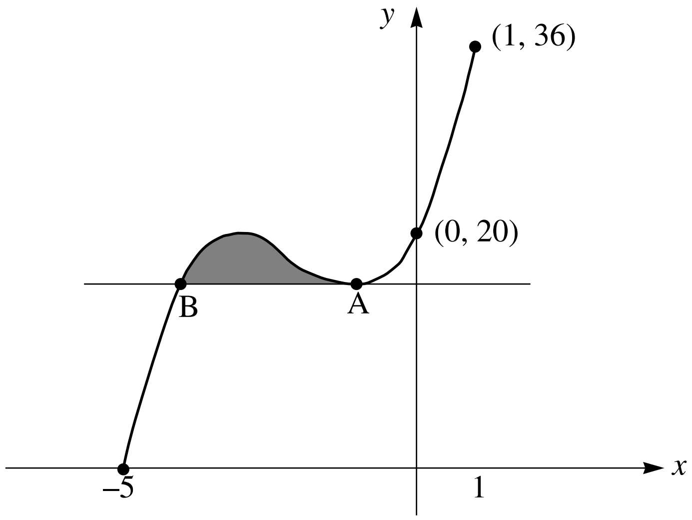
Find the derivative of (i) $f(x) = \frac{\ln(x)}{x}$.

(ii) $g(x) = \sin(\cos x)$.

(iii) $h(x) = \left(\frac{1+x}{1-x}\right)^2$.

Question 7

Consider the graph of $f : [-5, 1] \mapsto \mathbb{R}$, where $f(x) = x^3 + 6x^2 + 9x + 20$.



(a) (i) Find $f'(x)$.
(ii) Find the two values of x where the tangent to the graph of $f(x)$ is horizontal.

(b) (i) Expand $(x + 1)^2(x + 4)$.
(ii) Find where the tangent line to $f(x)$ at A extends to meet at B.
(iii) Find the area of the shaded region shown.

Question 8

Find (a) $\int_1^2 (4x^3 - 3)dx$.

(b) $\int_0^1 \left(e^{2x} + \frac{1}{x+1}\right)dx$.

Question 9

The slope at any point (x, y) on a curve C is given by $\frac{dy}{dx} = 3kx^2 + 2x - 2, k \in \mathbb{R}$. The area enclosed by C, the x-axis and the lines $x = 0$ and $x = 1$ is 2 sq. units. If the curve passes through the point (1, 3), find the value of k.

Question 10

(a) Show that $\int_{a}^{2a} \sqrt{1 + \frac{x}{a}}\,dx = \frac{2}{3}a(3\sqrt{3} - 2\sqrt{2})$.

(b) (i) Find the derivative of $f(x) = 2ax\sqrt{\frac{a+x}{a}}$.

(ii) **Hence** find $\int_{a}^{2a} x\sqrt{\frac{a}{a+x}}\,dx$.

Question 11

Consider the integrals $A_n = \int_{0}^{\frac{\pi}{2}} x^n \cos^2 x\,dx$ and $B_n = \int_{0}^{\frac{\pi}{2}} x^n \sin^2 x\,dx$.

(a) Evaluate $A_2 + B_2$.

(b) (i) Differentiate $x\sin 2x$.

(ii) **Hence** evaluate $A_1 - B_1$.

Question 12

Consider the integral $I_n = \int_{0}^{1} \frac{t^n}{(1+t)^n}\,dt$, $n \in \mathbb{Z}^+$.

(a) (i) Show that $\frac{t}{t+1} = 1 - \frac{1}{t+1}$.

(ii) Evaluate I_1.

(b) (i) Differentiate $\ln(t^2 + 2t + 1)$.

(ii) Show that $\frac{t^2}{(t+1)^2} = 1 - \frac{2t+1}{(t+1)^2}$.

(iii) **Hence** evaluate I_2.

Question 13

(a) Evaluate $\int_{2}^{4} \frac{x^3 - 4x^2 + x}{x^2}\,dx$.

(b) Evaluate $\int_{0}^{\frac{\pi}{4}} (\sin x + \cos x)^2\,dx$.

Question 14

Given that $\int_0^a f(t)dt = 5$ and $\int_0^a g(t)dt = -2$, find

(a) (i) $\int_0^a (f(t) - g(t))dt$.

(ii) $\int_0^a (3f(t) + 1)dt$.

(b) the value of k so that $\int_0^a (k + 2g(t))dt = 4$.

Question 15

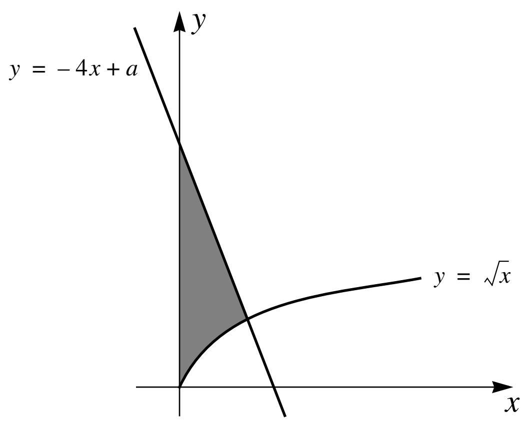
An object moving in a straight line has its displacement from an origin, O, described by the displacement equation $s(t) = \frac{1}{t^2} - \frac{2}{t} + 5, t > 0$.

(a) Find the object's (i) position after 1 second.
(ii) speed after 2 seconds.

(b) When will the object first be stationary after it starts its motion?

Question 16

A normal to the graph of $y = \sqrt{x}$ has equation $y = -4x + a, a \in \mathbb{R}$.



(a) Find a.

(b) Find the area of the shaded region shown in the diagram above.

Question 17

A particle is moving along the x-axis in such a way that its displacement from the origin, O, at time t is given by the equation

$$x(t) = 2t + \sin(2t), t \geq 0.$$

(a) Find the particle's initial (i) velocity.
(ii) acceleration.

(b) Find the value of t when the particle is stationary for the first time.

Question 18

Given that $f''(x) = -6x + 4\sin(x)$, $f(0) = 1$, and $f'(0) = 2$, find $f(1)$.

Question 19

An object starts from rest and for the first minute has an acceleration given by

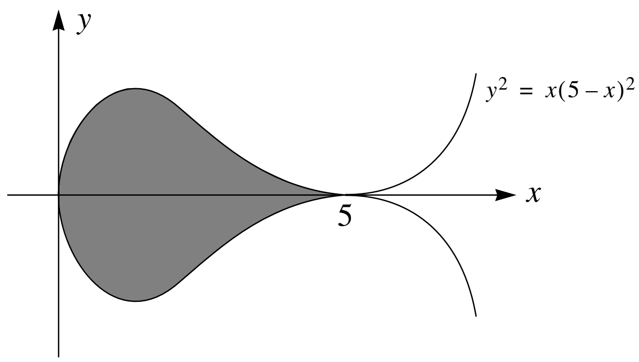
$$a(t) = \left(k - \frac{1}{10}t\right) \text{ m s}^{-1}, t \geq 0, k \in \mathbb{R}.$$

(a) After one minute it has a velocity of 22 m s^{-1}. Find the value of k.

(b) (i) When will the object next be stationary?
(ii) What is the object's displacement after the first minute of motion?

Question 20

(a) The area enclosed by the loop of the curve $y^2 = x(5-x)^2$ is shown by the shaded region in the diagram below. Find this area.



(b) The area of the shaded region shown is rotated about the x-axis, find the volume of the solid formed.

Question 21

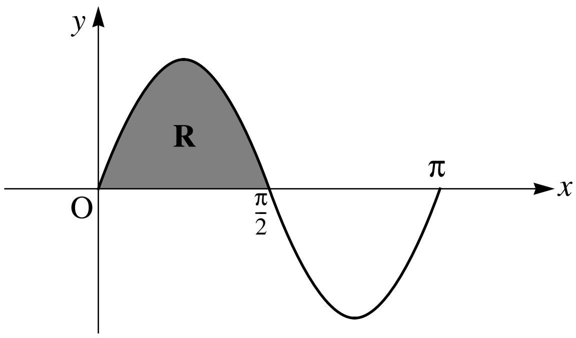
(a) Given that $f(x) = x^3 - 2x^2 - 4x$, find $f''(x)$.

(b) (i) Find the x-coordinate of the stationary points of $f(x)$.
(ii) Identify the points in (i) as either a local maximum or a local minimum, giving a clear reason for your choice.

(c) Find the x-coordinate(s) of the point(s) of inflexion of $f(x)$.

Question 22

Consider the shaded region, **R**, enclosed by part of the curve $y = \sin 2x$ and the x-axis over the interval $0 \le x \le \frac{\pi}{2}$.

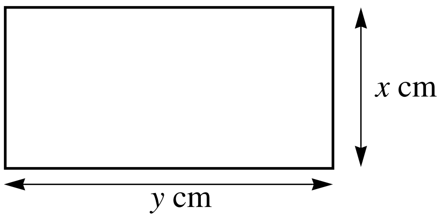


(a) Find the area of **R**.

(b) If the region **R** is rotated about the x-axis, find the volume of the solid generated.

Question 23

The rectangle of area A cm^2 and dimensions x cm by y cm has a constant perimeter of 20 cm.

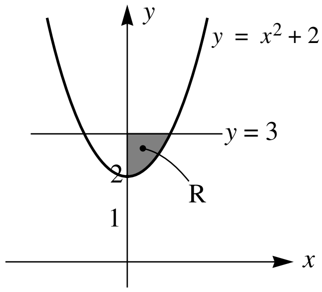


(a) Show that (i) $y = 10 - x$.
(ii) $A = 10x - x^2$.

(b) (i) Find $\frac{dA}{dx}$
(ii) Hence find the maximum area and justify your answer.

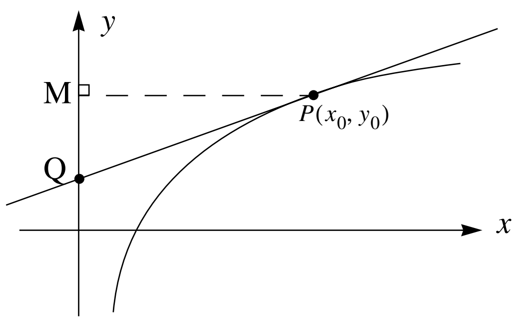
Question 24

The area of the shaded region, R, is rotated about the x-axis. Find the volume of the solid generated.



Question 25

P is a point on the curve with equation $y = b\ln\left(\frac{x}{a}\right), x > 0$ and where a and b are positive real constants. M is the foot of the perpendicular from P to the y-axis. The tangent at P cuts the y-axis at Q.



(a) Show that the gradient at P is $\frac{b}{x_0}$.

(b) Find the equation of the tangent at P.

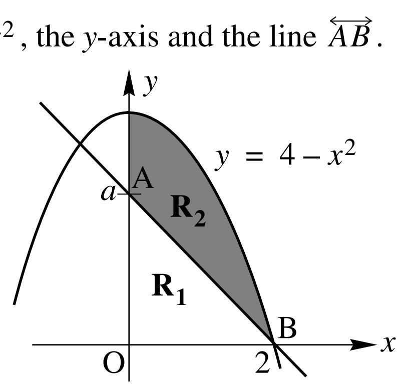
(c) Show that MP is constant.

Question 26

In the diagram shown, R_1 represents the region enclosed by the triangle OAB and R_2 represents the shaded region enclosed by the curve $y = 4 - x^2$, the y-axis and the line $\overleftrightarrow{AB}$.

(a) Find, in terms of a, the area of R_1.

(b) Given that the area of R_1 is equal to the area of R_2, find the value of a.



Question 27

The position of a particle along a straight line at time t is given by

$$x(t) = a\sin t - at\cos t,\ a \in \mathbb{R}^{+}.$$

(a) Find the acceleration of the object when
  (i) $t = 0$.
  (ii) $t = \pi$.

(b) If the particle experiences a zero acceleration at time $t = a$, where $a \neq \frac{2n+1}{2}\pi$, $n \in \mathbb{Z}^{+}$, show that $\tan(a) + a = 0$.

Question 28

(a) If $f(x) = x^3\cos(2x)$, find (i) $f'(x)$.
  (ii) $f''(x)$.

(b) If $g(x) = \frac{e^{(3x+1)}}{2x+1}$, find $g'(x)$.

Question 29

The functions $f(x)$ and $g(x)$ are both differentiable at $x = 0$. Given that $f(0) = f'(0) = -2$, $f''(0) = 4$ and that $g(x) = [f(x)]^3$, find

(a) $g'(0)$.

(b) $g''(0)$.

Question 30

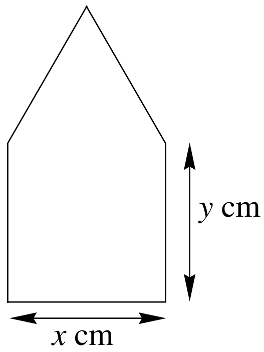
A scale model of the coastlines of two islands, C_1 and C_2, are approximated by the curves $y_1 = x^2 - 6x + 15$ and $y_2 = -(x-3)(x-5)$ for $0 \leq x \leq 5$ respectively.

(a) Sketch both coastlines on the same set of *xy*–axes.

(b) A rope, parallel to the *y*-axis is to run from C_1 to C_2. What is the length of the shortest rope needed (for this scale model).

Question 31

A window consists of a rectangular base surmounted by an aquilateral triangle as shown below.



The intensity of light, I units, passing through this window is proportional to its surface area, A m^2. The perimeter of the window is fixed at 12 m.

(a) Show that $y = 6 - \frac{3}{2}x$.

(b) Find the dimensions of the window that will allow the maximum intensity.

Question 32

(a) If $h(x) = \sin^2(\sqrt{x})$, find $h'(1)$.

(b) If $e^{-a} = \frac{1}{2}$, find $f'(a)$ given that $f(t) = \arcsin(e^{-t})$.

Question 33

Find the equation of the tangent to the curve with equation $y^3 = 2xy - x^2$ at the point (1, 1).

Question 34

Find the equation of the normal to the curve with equation $y\sin y = x\sin x$ at the point (π, π).

Question 35

(a) On the same set of axes, sketch the curves $y = 2 - x^2$ and $y = |x|$.

(b) Find the area of the region enclosed by both curves.

(c) The region in part (b) is now rotated about the y-axis. Find the volume of the solid of revolution formed.

Question 36

An object whose displacement from an origin O is s metres, moves in a straight line. The object's acceleration, a ms^{-2}, is given by $a = \frac{s^2}{s+1}$.

(a) (i) Using the fact that $a = \frac{dv}{dt}$, show that $a = v \cdot \frac{dv}{ds}$, where v is the object's velocity.

(ii) Hence find an expression for v^2 in terms of s given that $s = 0$ when $v = 2$.

(b) Given that $v > 0$ and that $v = \sqrt{k + \ln(2)}$ when $s = 1$, find the value of k.

Question 37

A curve C is defined implicitly by the equation $e^{\sin y} = \sin x + y^2$.

(a) Find the equation of the tangent to C at the point where $y = 0$ and $0 < x < \pi$.

(b) Find the equation of the normal to C at the point where $y = 0$ and $0 < x < \pi$.

Question 38

The function g is defined by $g(x) = (\log_2 x)^2 - x^2, x > 0$. If $g''(a) = 0$ has as its solution $(a\ln 2)^2 + (\ln a) = b$, find the value of b.

Question 39

Consider the curve with equation $y = \log_2 x$.

(a) Find the equation of the tangent at the point where

(i) $x = \frac{1}{2}$.

(ii) the slope is $\frac{1}{\ln 2}$.

(b) If $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{k}{x^2}$, find the value of k.

Question 40

(a) Find $\int_1^e \frac{1 + \ln x}{x} dx$.

(b) By using the substitution $u = \sqrt{x}$, find $\int_0^4 \frac{\sqrt{x}}{1 + x} dx$.

Question 41

Find the value of (a) $\int_{-1}^{1} |x^3 - 1| dx$.

(b) $\int_{-1}^{2} |x^3 - 1| dx$.

Question 42

Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}} dx$.

Question 43

(a) If $2\sin\theta = x$, find an expression for $\cos\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

(b) Find (i) $\int \sqrt{4 - x^2} dx$, $|x| \le 2$.

(ii) $\int x\sqrt{4 - x^2} dx$, $|x| \le 2$.

Question 44

The function g is defined by $g(x) = xe^{-x}$.

(a) Find (i) the stationary point of g.
(ii) the point of inflection of g.

(b) Sketch the curve of g.

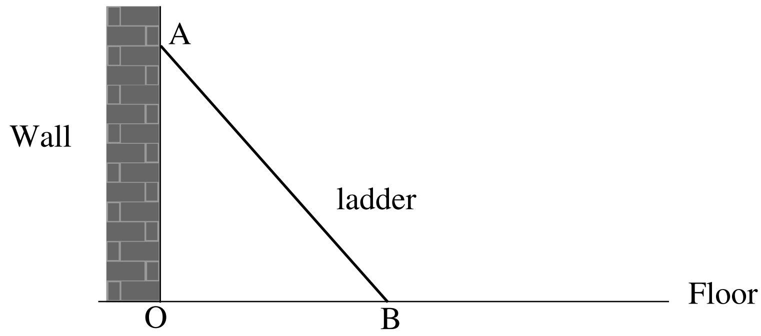
(c) Determine the values of k for which the equation $k = xe^{-x}$ has
(i) two real roots.
(ii) one real root.
(iii) no real roots.

Question 45

The volume of a cube is increasing at $20\ \text{cm}^3\text{s}^{-1}$. At what rate are the lengths of the edges of the cube increasing when its volume is $8000\ \text{cm}^3$.

Question 46

A ladder $[AB]$, 16 m long, is leaning against a vertical wall. The end resting on a horizontal floor is sliding away from the wall at a rate of $\sqrt{3}$ m min^{-1}.



(a) If OA = y m and OB = x m, find an equation connecting x and y.

(b) Find the rate at which point A is moving toward the floor when point B is 8 m from the wall.

Question 47

A particle moving along a straight line from a fixed point O at time t seconds has its velocity, v ms^{-1}, given by

$$v(t) = 4te^{-\frac{t}{2}}, t \geq 0.$$

(a) (i) Find the maximum velocity of this particle and justify that this is the maximum velocity.
(ii) Find the particle's minimum acceleration.

(b) Find the distance travelled by the particle after of being in motion for 2 seconds.

Question 48

(a) The function $g(x) = \dfrac{\arccos(x)}{f(x)}$, where $f\left(\frac{1}{2}\right) = 1$, $f'\left(\frac{1}{2}\right) = \dfrac{\sqrt{3}}{\pi}$. Find $g'\left(\frac{1}{2}\right)$.

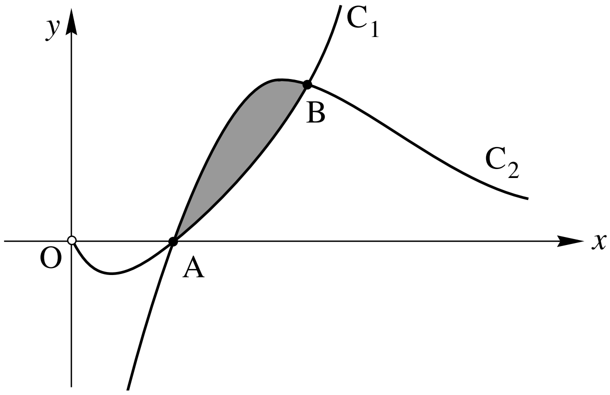
(b) Find $h'\left(\frac{\pi}{2}\right)$ given that $h(x) = \arctan(e^{\cos x})$.

Question 49

(a) Find (i) $\int \frac{\ln x}{x} dx$.

(ii) $\int x \ln x dx$.

Parts of the graphs of $f(x) = \frac{9\ln x}{x}, x > 0$ and $g(x) = x\ln x, x > 0$ are shown below.



(b) (i) Identify which of f or g correspond to C_1 and which of f or g correspond to C_2.

(ii) Find the coordinates of A and B.

(iii) Find the area of the shaded region show above.

Question 50

Find the particular solution to the differential equation $xy\frac{dy}{dx} = 1 - x^2$ given that $y(1) = -1$.

Question 51

Find the expression for y in terms of x given that $2\frac{dy}{dx} = \frac{y}{1+x^2}, y(0) = 1$.

Question 52

(a) Simplify $\frac{1}{y-1} - \frac{1}{y}$.

(b) Find an expression for y in terms of x given that x and y satisfy the differential equation

$$x\frac{dy}{dx} + y = y^2, y(1) = 2 \text{ and } x > 0 \text{ and } y > 1.$$

(c) Sketch the graph of the solution curve in (b).

Question 53

During a chemical reaction, the amount, R kg, of chemical formed at time t hours is modelled by the differential equation

$$\frac{dR}{dt} = 4 - \frac{R}{15}.$$

Initially there are 10 kg of the chemical in the solution.

(a) Find an expresion for R in terms of t.

(b) How long will it take for 20 kg of the chemical to form?

ALGEBRA

1. (a) (i) 3 (ii) 50 (b) 798
2. (a) (i) 3 (ii) 5 (b) $a = \frac{5}{2}, b = 3^{10}$
3. (a) $u_1 = -8, u_2 = -17, u_3 = -26$ (b) –485
4. (a) ±2 (b) 3 (c) 2.5 (d) 0.1
5. $a = 1, b = 1$
6. (a) 8 (b) 0, 8 (c) 20 (d) $\pm 2\sqrt{2}$ (e) $2\sqrt{2}$
7. (a) (i) 7.5 (ii) 1 (b) $\frac{1 \pm \sqrt{13}}{2}$
8. (a) (i) $(x-5)(x-1)$ (ii) $x < 1$ or $x > 5$
   (b) (i) $x < 6$ (ii) $x > 4$
   (c) $4 + \frac{1}{2}\sqrt{6}$
9. (a) (i) -1 or $-1 \pm \sqrt{2}$ (b) $e^{-1}, e^{-1 \pm \sqrt{2}}$
10. (a) (i) $x^2 + (1+a)x + a$ (ii) $(x+2)(x+1)$
    (b) (i) $1, e$ (ii) 0, 1
11. (a) 2 (b) (i) 45 (ii) 225
12. (a) $\sqrt{2}$ (b) $\frac{1}{2}(\sqrt{6} + 4 + \sqrt{2} + 2\sqrt{3})$
13. $a = 1, b = 11$ and $c = -\frac{1}{2}$ [or $a = \frac{1}{2}, b = 12, c = -\frac{1}{2}$ if $S_{12} = \frac{1}{2} \times 2^{12} - \frac{1}{2}$]
14. $r = \frac{-1 + \sqrt{5}}{2}$
15. (a) $u_1 = p + q - 1$ (b) $S_{p+q} = \frac{1}{2}(p+q)(p+q-1)$
16. (a) –2, 1 (b) e, e^{-2} (c) 0
17. (a) $-2a^2 + 5a + 3$ (b) (i) 3 (ii) 1
18. (i) $\frac{1}{2}(8+a)$ (ii) 50 (iii) $\frac{9}{8}$
19. (a) $a = \frac{b^2}{b-1}, b > 1$ (b) 1
20. $x = \frac{13}{4}, y = \frac{7}{4}$
22. (a) 144 (b) $x = 18, y = 8$ (c) (i) $\frac{2}{3}$ (ii) 54
23. $\alpha = 32, \beta = 2$ or $\alpha = 2, \beta = 32$
24. (a) 120 (b) (i) 3 (ii) 5 (c) (i) $u_r = 2r + 1$ (ii) 41
25. (a) $a = 1, b = -1$
26. (b) 2
27. $x = -1, a = 1, b = 5$

28. (a) $\frac{12}{65}$ (b) $xy = 1$

29. 1, 2

31. $x < \log_2 3$

32. $2^n \cdot \frac{(2n)!}{(n!)^2}$

33. (b) 8th, 9th and 10th terms

34. $3 + i, -1$

35. $a = 2, b = 1, c = \sqrt{3}$

36. $z = 1 \cdot e^{\frac{2\pi}{3}i}$

37. $a = 0, b = 0$

39. (a) $m = -2, n = 1.5$ (b) $z = \frac{5}{2} \cdot e^{-\arctan\left(\frac{3}{4}\right)i}$

40. (a) (i) $|w| = 2, \operatorname{Arg}(w) = \frac{2\pi}{3}$ (ii) $|z| = \sqrt{2}, \operatorname{Arg}(z) = -\frac{\pi}{4}$

    (iii) $|wz| = 2\sqrt{2}, \operatorname{Arg}(wz) = \frac{5\pi}{12}$

    (b) (i) $wz = (\sqrt{3} - 1) + (\sqrt{3} + 1)i$ (ii) $\frac{1}{4}(\sqrt{6} + \sqrt{2})$

41. $a = 4, b = -5$

42. 2

43. (a) (i) -16 (ii) see soln. (b) 4

    (c) $0, \sqrt{2}(1 + i), \sqrt{2}(-1 + i), -\sqrt{2}(1 + i), \sqrt{2}(1 - i)$

44. $3 - 2i, -3 + 2i$

45. $\sqrt[4]{2} \cdot e^{\frac{\pi}{12}i}, \sqrt[4]{2} \cdot e^{\frac{7\pi}{12}i}, \sqrt[4]{2} \cdot e^{-\frac{5\pi}{12}i}, e^{-\frac{11\pi}{12}i}$

46. (a) $a = 0, b = 1$ (b) $a = \frac{1}{\sqrt{2}}, b = \frac{1}{\sqrt{2}}$ or $a = -\frac{1}{\sqrt{2}}, b = -\frac{1}{\sqrt{2}}$

    (c) $a = 0, b = -1; a = \frac{\sqrt{3}}{2}, b = \frac{1}{2}; a = -\frac{\sqrt{3}}{2}, b = \frac{1}{2}$

FUNCTIONS AND EQUATIONS

1. (a) (i) 1 (ii) $\frac{1}{3}$ (iii) $\frac{1}{6}\sqrt{6}$ (b) $x > 3$ (c) See soln.
2. (a) (i) $\mathbb{R}\backslash\{1\}$ (ii) $\mathbb{R}$ (b) (ii) $(g \circ f)(x) = \frac{1}{(x-1)^3}, x \neq 1$
3. (a) (i) 2 (ii) 0 (b) $x \leq 7$ (c) (i) $]-\infty,7[$ (ii) $[0, \infty[$
4. $a = 0, b = 1$
5. (a) $f(x) = 3(x-1)^2 + 4$ (b) (i) $(1, 4)$ (ii) 1
   (c) (i) $(0, 7)$ (ii) See soln.
6. (a) $]-1, \infty[$ (b) (ii) $(f \circ g)(x) = \ln(e^x + 1)$ (iii) $\mathbb{R}$
7. $a = 3, b = 11, c = 2$
8. (a) (i) 2.5 (ii) 1 (b) $f(x) = x^4 + 3x^2 + 1$
9. (a) $(f \circ g)(x) = xe^x, x \in \mathbb{R}$ (b) 0, ln2 (c) ln2
10. (a) $f^{-1}(x) = \frac{1}{x} + 2, x \in\ ]0,\infty[$ (b) See soln. (c) $(1 + \sqrt{2}, 1 + \sqrt{2})$
11. (a) (i) $-\ln 2$ (ii) $-\ln 2$ (b) $a > 1$
12. (a) See soln. (b) $[2, \infty[$ (c) (i) $(f \circ g)(x) = x - 6, x \geq 2$ (ii) $[-4, \infty[$
13. (a) $a = 1, b = 1 + e$ (b) $f^{-1}(x) = 1 + e^{x+1}, x \in \mathbb{R}$ (c) See soln.
14. $x > 1, x \neq e$
15. (a) See soln. (b) (i) $\left\{2, -\frac{1}{3}\right\}$ (ii) $\left\{x \mid -\frac{1}{3} \leq x \leq 2\right\}$
16. (a) $\left\{x \mid x > -\frac{2}{3}\right\}$ (b) (i) $f^{-1}(x) = 3^{x-1} - \frac{2}{3}, x \in \mathbb{R}$ (ii) See soln.
    (iii) $f^{-1}(x) = \frac{1}{3}(g(x) - 2)$
17. See soln.
18. (a) (i) $g(x) = f(x-3) + 2$ (ii) $g(x) = \frac{1}{x-3} + 2, x \neq 3$ (b) See soln
19. $a = 2, b = -7$
20. (a) See soln (b) (i) $g(x) = f(x+1) + 2$ (ii) $g(x) = (x+1)^2 + 2$
    (c) (i) $\varnothing$ (ii) 3 (iii) See soln
21. (a) 0 (b) 0
22. (a) $g(x) = f(x-a) - b$ (b) $a = 0, b = -4$
23. (a) $2e + 3$ (b) ln2
24. (a) 0 (b) 7
25. (a) 1 (b) See soln
26. $]-\infty,0[\ \cup\ ]8,\infty[$
27. (a) See soln (b) (i) 0^+ (ii) 0^+
    (c) (i) $\mathbb{R}$ (ii) See soln (iii) $]0, 4]$
28. (a) See soln (b) $a = 1$ (c) See soln (d) See soln
29. (a) (i) See soln (ii) 1, –2 (iii) 1, –2 (b) $x < -2$ or $x > 1$

30. (a) (i) $f(x) = \frac{1}{2} - \frac{3}{2x}, g(x) = -\frac{1}{2} + \frac{3}{2x}$ (ii) See soln
    (b) $x < 0$ or $x > 1$
31. (a) (i) See soln (ii) $x < 1$ (b) $\{x|1 < x < \sqrt{7}\} \cup \{x|x < -1\}$
32. (a) $a = 2, b = -7$ (b) $\left\{x|x < \frac{1}{2}, x \neq 0\right\}$
33. (b) 0.25
34. $a = 1.25$
35. (a) See soln (b) (i) $x < -3$ or $1 < x < 3$ (ii) $-1 < x < 0$ or $x > 3$
36. $a = \frac{4}{\pi + 2}, b = \frac{2\pi}{\pi + 2}$
37. $\{x|0 < x < 2\}$

CIRCULAR FUNCTIONS AND TRIGONOMETRY

1. (a) $\frac{4}{5}$ (b) $\frac{24}{25}$ (c) $\frac{4}{3}$
2. $\sin^2\theta < \sin\theta < \frac{1}{\sin\theta}$
3. (a) $\frac{\sqrt{9-a^2}}{3}$ (b) 36° (c) $\frac{\sqrt{3}}{4}$ (d) $-\cot\theta$
4. (a) $-\frac{2}{\sin^2\theta}$ (b) $30^\circ, 150^\circ, 210^\circ, 330^\circ$
5. (a) 0 (b) (i) $0, \frac{\pi}{2}, \pi, 2\pi$ (ii) $\frac{3\pi}{2}$
6. (a) $a = \pi, b = 2, c = 4\pi$ (b) $\frac{2\pi}{3}, \frac{4\pi}{3}$
7. $a = \frac{3}{2}, b = 2, c = \frac{7}{2}$
8. (a) $[-2, 2]$ (b) (i) $\frac{1}{7}$ (ii) $\frac{4}{13}$
9. (a) $\frac{\pi}{2}, \frac{3\pi}{2}$ (b) 4
10. (a) $2x\sin\theta$ (b) 6
11. 10
12. (a) (i) $\frac{1}{4}\pi r^2$ (ii) $\frac{1}{16}\pi r^2$ (iii) $\frac{1}{64}\pi r^2$ (b) 5 (c) 9
13. $b = \sqrt{3} + 1$
14. $\frac{1}{2}(3 - \sqrt{3})$
15. 31
16. 3
17. (a) 60° (b) (i) $\frac{2\pi}{3}$ cm (ii) $\frac{2\pi}{3}$ cm^2 (c) 2 cm^2
18. (a) 54° (b) 45° (c) 30°
19. (a) $\frac{4a}{5-a}$ (b) $\frac{\pi}{4}$
20. (a) 4 (b) $8\sqrt{3}$ cm^2
21. (a) 8 cm (b) $\frac{16}{3}(3\sqrt{3} - \pi)$ cm^2
22. $a = -4, b = \frac{2}{3}, c = 6$
23. (a) (i) 0.4 (ii) $a = 2, k = 5\pi, c = 1$ (b) $\frac{\pi}{12}$ (c) $n < -m$
    (d) See soln (e) 4π
24. (a) 28 dB (b) 26 dB (c) 0.25 sec (d) 1.5 sec
25. (a) 3000 (b) 2050 (c) See soln (d) 20/3 yrs

26. $\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}$

27. (a) $\cos\theta + \sin\theta$ (b) $\pi, \frac{3\pi}{4}$

28. (a) (i) $\frac{\sqrt{7}}{4}$ (ii) $-\frac{5}{8}$ (b) $\frac{\sqrt{6}+\sqrt{182}}{16}$

29. (a) (i) 6 (ii) $a = 8, b = -1$ (b) 8

30. (b) $\frac{\pi}{6}, \frac{5\pi}{6}$

31. $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{3\pi}{2}\right\}$

32. $a = 2 = b$

33. (b) $\frac{-1+\sqrt{5}}{4}$

35. $x = \frac{\pi}{3}, y = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}, y = \frac{2\pi}{3}$ or $x = \frac{\pi}{6}, y = \frac{\pi}{3}$ or $x = \frac{2\pi}{3}, y = \frac{5\pi}{6}$

36. 2

37. (a) $\frac{4\pi}{3} - 2\ \text{cm}^2$ (b) $\frac{2\pi}{3}$ from O.

38. $x = a, y = b$

39. (a) (i) $2x$ m (ii) $2\sqrt{3}x$ m (ii) $\sqrt{10}x$ m (b) $a = 1, b = 13, c = 4$

40. (a) (i) $[-1, 3]$ (ii) See soln

    (b) $f^{-1}(x) = \arcsin\left(\frac{x-1}{2}\right) + \frac{\pi}{2}, x \in [-1, 3]$

    (c) See soln

41. (a) (i) See soln (ii) See soln (iii) $\frac{\pi}{2}$ (b) $-\frac{7}{25}$

MATRICES

1. (a) (i) 9 (ii) $\frac{1}{3}\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$ (b) $X = \begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}$

2. (a) (i) $\begin{bmatrix} 14 & 4 \\ 0 & -8 \end{bmatrix}$ (ii) $\begin{bmatrix} 26 & -28 \\ 39 & 21 \end{bmatrix}$ (b) $x = 2, y = -1$

3. (a) $\begin{bmatrix} 13 \end{bmatrix}$ (b) not possible (c) not possible (d) $\begin{bmatrix} 1 \\ 10 \end{bmatrix}$

   (e) not possible

4. (i) 3 (ii) 2

5. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) (i) $\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$ (ii) $x = -1, y = 13, z = 7$

6. $x = 10, y = 2$

7. (a) $\frac{1}{10}\begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix}$ (b) (1.2, 1.4)

8. (a) (i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} \cos 2\theta & -2\cos\theta \\ 2\sin\theta & -\cos 2\theta \end{bmatrix}$ (b) $\frac{3\pi}{4}$

9. (a) $\begin{bmatrix} 8 & 10 \\ 0 & 8 \end{bmatrix}$ (b) (i) $\begin{bmatrix} 8 & 10 \\ 0 & 8 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

10. 2 or –1

11. (a) (i) –2 (ii) $-\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -7 & 3 \end{bmatrix}$ (b) $x = 8, y = -13$

12. Solution of the form $\alpha, \beta = -\alpha, \gamma = 3\alpha$ where $\alpha \in \mathbb{R}$ (e.g., $\alpha = 1, \beta = -1, \gamma = 3$)

13. (a) (i) $2I$ (ii) $\frac{1}{2}B$ (b) $x = 4.5, y = 4, z = 3.5$

14. $x = 2 + \sqrt{2}, 2 - \sqrt{2}, 2$

15. (a) $a - 2$ (b) (i) $x = 2, y = -2, z = 0$ (ii) $\mathbb{R}\backslash\{2\}$

16. (a) $k = 3$ (b) $x = 3\lambda - 3, y = -4\lambda + 5, z = \lambda, \lambda \in \mathbb{R}$

17. (a) (i) $\Delta = 25 - p^2$ (ii) ±5 (b) $\mathbb{R}\backslash\{\pm 5\}$

    (c) When $p = 5$, $x = 2 - 9k, y = k, z = 8k$, $k \in \mathbb{R}$

18. (b) (i) $x = -1, y = -2, z = 3$

VECTORS

1. (a) (i) $2\boldsymbol{i} + 4\boldsymbol{j}$ (ii) $8\boldsymbol{i} - \boldsymbol{j}$ (b) (i) 12 (ii) $\frac{36}{325}$

2. 2

3. (a) $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$ (b) 1 (c) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (d) $\sqrt{34}$

4. (a) 9 (ii) $\frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ (b) $x = \frac{1}{2}, y = \frac{3}{4}$

5. (a) (i) 7 (ii) 3 (b) $7\sqrt{2}$ (c) 40

6. $\alpha = 2, \beta = -1, \gamma = 2$

7. $\frac{3}{8}$

8. $\frac{1}{3}, -1$

9. (a) 7 (b) 25 (c) 0

10. (a) $r_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$ (b) $\frac{5}{2\sqrt{33}}$

11. (a) (i) $\begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$ (ii) $\sqrt{34}$ (b) 2

12. (a) (i) $\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$ (ii) $\begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}$ (iii) $\begin{pmatrix} 2 \\ 4 \\ -4 \end{pmatrix}$ (b) $\frac{13}{3\sqrt{21}}$

    (c) $-10\boldsymbol{i} - 5\boldsymbol{j} - 15\boldsymbol{k}$

    (d) (i) $2x + y + 3z = 14$ (ii) $-\frac{1}{\sqrt{14}}(2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})$

    (e) $z = \frac{1-y}{2} = 13 - 2x$

13. (a) 19 (b) $\frac{19}{2\sqrt{259}}$

14. (a) (7, 9, 0) (b) $\boldsymbol{r} = \frac{1}{4}\begin{pmatrix} 1 \\ 0 \\ -9 \end{pmatrix} + \lambda\begin{pmatrix} 3/4 \\ 1 \\ 1/4 \end{pmatrix}$ or $\boldsymbol{r} = \frac{1}{4}\begin{pmatrix} 1 \\ 0 \\ -9 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}, \lambda, \mu \in \mathbb{R}$

    (c) $\dfrac{x - \frac{1}{4}}{\frac{3}{4}} = \dfrac{y}{1} = \dfrac{z + \frac{9}{4}}{\frac{1}{4}}$

15. (a) $\dfrac{1}{\sqrt{5}}(2\boldsymbol{i} + \boldsymbol{j})$ (b) $\dfrac{4\sqrt{5}}{21}$
16. 3
17. (a) $\alpha = -\dfrac{1}{3}, \beta = \dfrac{1}{2}$ (b) $\dfrac{3}{2}\sqrt{38}$ sq. units
18. $2x - y + 3z = -1$
19. $\boldsymbol{r} \bullet \boldsymbol{n} = -5$
20. (a) $\hat{\boldsymbol{n}}_1 = \dfrac{1}{\sqrt{6}}(-\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}), \hat{\boldsymbol{n}}_2 = \dfrac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$ (b) $\dfrac{1}{\sqrt{6}}$
21. (a) –36 (b) (i) $\begin{pmatrix} -3 \\ -1 \\ 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix}$ (c) $5x - 11y + z = -18$
22. $\left(\dfrac{17}{15}, -\dfrac{27}{15}, \dfrac{13}{15}\right)$
23. $a = 3, b = 29$
24. (a) $r = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}, \lambda, \mu \in \mathbb{R}$ (b) $3x - 11y + 7z = -21$

STATISTICS AND PROBABILITY

1. (a) 0.5 (b) (i) 56.60 (ii) 3.5
2. 0.055
3. (a) a (b) $1 - 2a$
4. (a) 0.07 (b) 0.86 (c) $\frac{86}{93}$
5. (a) See soln (b) (i) 4.6 (ii) 3 (iii) 4
6. (a) x (b) $x + 1$ (c) $46\frac{1}{3}$
7. (a) $\frac{9}{22}$ (b) $\frac{1}{22}$
8. (a) (i) 3 yrs (ii) 3.8 (iii) 1.8 (b) $\frac{11}{14}$
9. (a) $\frac{11}{20}$ (b) (i) 2.8 (ii) 0.86 (c) (i) $\frac{9}{400}$ (ii) $\frac{7}{40}$
10. (a) 0.92 (b) 90 (c) 63
11. (a) $\frac{8}{125}$ (b) $\frac{36}{125}$
12. (a) $\frac{1}{2}x + \frac{1}{8}$ (b) (i) $\frac{16x^2}{16x^2 + 8x + 1}$
13. (a) (i) $3x$ (ii) $4x$ (b) $\frac{1}{3}$ (c) $\frac{20}{27}$
14. (a) $\frac{8}{9}$ (b) (i) $\frac{8}{3}$ (ii) $\frac{4}{3}$
15. (a) 24 (b) 336
16. See soln
17. (a) 0.06 (b) 0.21 (c) 0.44 (d) 0.79 (e) $\frac{9}{44}$
18. (a) (i) $\frac{5}{32}$ (ii) $\frac{3}{16}$ (iii) $\frac{31}{32}$ (b) 2.5
19. $a = 8, b = 11$
20. $a = 18, b = 6$
21. See soln
22. $a = 25, b = 6$
23. $\frac{n+1}{2^{n-1}}$
24. (a) $\frac{3}{8}$ (b) $\frac{1}{8}$ (c) $\frac{1}{4}$
25. (a) 0.5 (b) See soln (c) (i) $\frac{2-\sqrt{2}}{4}$ (ii) 0.25 (d) $\frac{\pi}{2}$ (e) $\frac{\pi}{2}$
26. (a) $\frac{4}{\pi}$ (b) $\frac{2}{3}$ (c) b = 4 (d) (i) $F(x) = \begin{cases} 0, & x < 0 \\ \frac{4}{\pi}\arctan(x), & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$

(ii) See soln (iii) $c = 8$

27. (a) (i) $F(x) = \begin{cases} 0, & x < 0 \\ 1 - \dfrac{1}{x+1}, & x \geq 0 \end{cases}$ (ii) See soln (b) $c_q = \dfrac{1}{1-q} - 1$

(c) (i) 1 (ii) $\dfrac{8}{3}$ (iii) 19

28. $\dfrac{408}{2187}$

CALCULUS

1. $y = 2x$
2. (a) (i) $\frac{1}{2}$ (ii) 2 (b) $\frac{27}{8}\sqrt{2}$
3. See soln
4. (a) $\frac{1}{\sqrt{3}} + \frac{2\pi}{9}$ (b) $y = \left(1 + \frac{\pi}{2}\right)x - \frac{\pi^2}{8}$
5. –2
6. (a) (i) $\frac{1 - \ln(x)}{x^2}$ (ii) $(\sin x)(\cos(\cos x))$ (iii) $\frac{4(1+x)}{(1-x)^3}$
7. (a) (i) $3x^2 + 12x + 9$ (ii) –1, –3
   (b) (i) $x^3 + 6x^2 + 9x + 4$ (ii) (–4, 16) (iii) $\frac{27}{4}$ sq. units
8. (a) 12 (b) $\frac{1}{2}e^2 - \frac{1}{2} + \ln(2)$
9. $\frac{16}{9}$
10. (a) $\frac{2}{3}a(3\sqrt{3} - 2\sqrt{2})$ (b) (i) $2a\sqrt{\frac{a+x}{a}} + x\sqrt{\frac{a}{a+x}}$ (ii) $\frac{2}{3}a^2\sqrt{2}$
11. (a) $\frac{\pi^3}{24}$ (b) (i) $\sin(2x) + 2x\cos(2x)$ (ii) –1
12. (a) (i) $\frac{t}{t+1}$ (ii) $1 - \ln(2)$ (b) (i) $\frac{2t+2}{t^2+2t+1}$ (ii) $\frac{t^2}{(t+1)^2}$
    (iii) $1.5 - 2\ln(2)$
13. (a) $\ln(2) - 2$ (b) $\frac{\pi}{4} + \frac{1}{2}$
14. (a) 7 (b) $15 + a$ (b) $\frac{8}{a}$
15. (a) (i) 4 (ii) 0.25 (b) 1
16. (a) 18 (b) $\frac{104}{3}$ sq. units
17. (a) (i) 4 (ii) 0 (b) $\frac{\pi}{2}$
18. $6 - 4\sin(1)$
19. (a) $\frac{101}{30}$ (b) (i) $\frac{202}{3}$ (ii) 2460
20. (a) $\frac{40}{3}\sqrt{5}$ sq. units (b) $\frac{625}{12}\pi$ cubic units
21. (a) $6x - 4$ (b) (i) $-\frac{2}{3}, 2$ (ii) min at $x = 2$, max at $x = -\frac{2}{3}$ (c) $\frac{2}{3}$
22. (a) 1 sq. units (b) $\frac{\pi^2}{4}$ cubic units

23. (a) See soln (b) (i) $10 - 2x$ (ii) 25 sq. units
24. $\frac{52}{15}\pi$ cubic units
25. (a) $\frac{b}{x_0}$ (b) $y = \frac{b}{x_0}x - b + y_0$
26. (a) a sq units (b) $\frac{8}{3}$
27. (a) (i) 0 (ii) $-a\pi$
28. (a) (i) $3x^2\cos(2x) - 2x^3\sin(2x)$ (ii) $(6x - 4x^3)\cos(2x) - 12x^2\sin(2x)$
    (b) e
29. (a) –24 (b) 0
30. (a) See soln (b) 5.5 units
31. (b) $x = \frac{12}{33}(6 + \sqrt{3})$, $y = \frac{6}{11}(5 - \sqrt{3})$
32. (a) $\frac{1}{2}\sin 2$ (b) $-\frac{1}{\sqrt{3}}$
33. $y = 1$
34. $y = -x + 2\pi$
35. (a) See soln (b) $\frac{7}{3}$ sq. units (c) $\frac{5\pi}{6}$ cubic units
36. (a) (i) See soln (ii) $v^2 = s^2 - 2s + \ln(s+1) + 4$ (b) 3
37. (a) $y = 0$ (b) $x = \frac{\pi}{2}$
38. 1
39. (a) (i) $(\ln 2)y = 2x - 1 + \ln 2$ (ii) $(\ln 2)y = x - 1$ (b) $\frac{1 - \ln 2}{(\ln 2)^2}$
40. (a) 1.5 (b) 2(2 – arctan(2))
41. (a) 2 (b) $\frac{19}{4}$
42. $2 - \sqrt{2}$
43. (a) $\frac{\sqrt{4 - x^2}}{2}$ (b) (i) $2\arcsin\left(\frac{x}{2}\right) - \frac{x}{2}\sqrt{4 - x^2} + c$, $|x| \le 2$
    (ii) $-\frac{1}{3}\sqrt{(4 - x^2)^3} + c$, $|x| \le 2$
44. (a) (i) $(1, e^{-1})$ (ii) $(2, 2e^{-2})$ (b) See soln
    (c) (i) $0 < k < \frac{1}{e}$ (ii) $k \le 0$ or $k = \frac{1}{e}$ (iii) $k > \frac{1}{e}$
45. $\frac{1}{60}$ cm s^{-1}
46. 1 m s^{-1}
47. (a) (i) $8e^{-1}$ m s^{-1} (ii) $-4e^{-2}$ m s^{-2} (b) $16(1 - 2e^{-1})$ m
48. (a) $-\sqrt{3}$ (b) –0.5

49. (a) (i) $\frac{1}{2}(\ln x)^2 + c$ (ii) $\frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + c$

(b) (i) $f \leftrightarrow C_2$, $g \leftrightarrow C_1$ (ii) $A \equiv (1, 0)$, $B \equiv (3, \ln 3)$

(iii) $\frac{9}{2}(\ln 3)[(\ln 3) - 1] + 2$ sq. units

50. $y = -\sqrt{2\ln x - x^2 + 2}$

51. $y = e^{\frac{1}{2}\arctan(x)}$

52. (a) $\frac{1}{y(y-1)}$ (b) $y = \frac{2}{2-x}, x > 0, y > 1$ (c) See soln

53. (a) $R = 60 - 50e^{-\frac{1}{15}t}, t \geq 0$ (b) 15ln(1.25) hrs