BarCharts, Inc.®
WORLD'S #1 ACADEMIC OUTLINE
Quick Study® ACADEMIC

ALGEBRA - Part 2

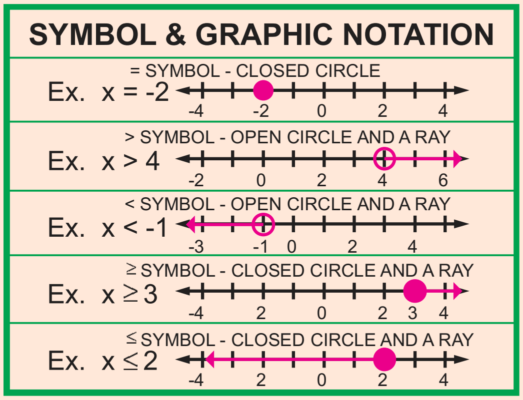
graphs, functions, conic sections & problem solving

PARTS 1 & 2 COMBINED COVER PRINCIPLES FOR BASIC, INTERMEDIATE AND COLLEGE COURSES

GRAPHING

REAL NUMBER LINE

*Chart of the graphs, **on the real number line**, of solutions to one-variable equations:*



* Direction of ray is determined by picking (at random) a value on each side of the circle. Ray goes in direction of the point which makes the inequality true.

- **ABSOLUTE VALUE STATEMENTS**
  1. **Equalities:** To solve $|ax+b|= c$, where $c > 0$, solve both equations $ax + b = c$ and $ax + b = -c$, and graph the union of the two solutions.
  2. **Inequalities:**
     a. To solve $|ax + b| < c$, where $c > 0$, solve $ax + b < c$ and $ax + b > -c$; these two inequalities may be written as one $-c < ax + b < c$; graph the **intersection** of the two solutions.
     b. To solve $|ax + b| > c$, where $c > 0$, solve $ax + b > c$ or $ax + b < -c$; graph the union of the two solutions.

RECTANGULAR (OR CARTESIAN) COORDINATE SYSTEM

*Method, using two perpendicular lines (intersecting at **90 degree** angles), for locating and naming points of a plane. The vertical line is the **y-axis**. The horizontal line is the **x-axis**. The point where they intersect is called the **origin**.*

- **LOCATING POINTS (ORDERED PAIRS)**

  Each point on coordinate plane is named or located by using an ordered pair of numbers separated by a comma and enclosed in a set of parenthesis; first number is **x**-coordinate or **abscissa**; second number is **y**-coordinate or **ordinate**; that is, an ordered pair is of the **form** (x,y). The **origin** is $(0,0)$.

- **QUADRANTS**

  The **x-axis** and the **y-axis** separate the plane into fourths. Each fourth is called a **quadrant**. The quadrants are labeled using Roman numerals, starting in the upper right section, and continuing counterclockwise through quadrants I, II, III, and IV (which is located in the lower right section).

- **DISTANCE FORMULA:** $d = \sqrt{(a-c)^2 + (b-d)^2}$

  Finds distance between two points, (a,b) and (c,d); is derived from the application of the Pythagorean Theorem and always results in a non-negative number.

- **MIDPOINT FORMULA:** $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

  Determines the coordinates of the midpoint of a line segment with endpoints of (x_1,y_1) and (x_2,y_2).

LINES

SLOPE OF A LINE

The slope of a line can loosely be described as the slant of the line. If the line slants up on the right end of the line then the slope will be a positive number. If the line slants up on the left end of the line then the slope will be a negative number. If the line is horizontal then the slope is zero. If the line is vertical then the line has no slope, it is undefined.

- **FORMULA:** If line is not vertical then slope (indicated by **m**) can be found using two distinct points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ of the line and using **x-coordinates** and **y-coordinates** in the formula:

$$m = \left(\frac{y_2-y_1}{x_2-x_1}\right) = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$$

- **PARALLEL:** The slopes of parallel lines are equal.
- **PERPENDICULAR:** The slopes of perpendicular lines are negative reciprocals. If the slope of L_1 is m_1 and the slope of L_2 is m_2, and the lines are perpendicular then $m_1 = -1/m_2$ or $(m_1)(m_2) = -1$. **EX:** If the slope of a line is $-^1/_2$ then the slope of the line which is perpendicular to it is $+2$.

LINEAR EQUATIONS (EQUATIONS OF LINES)

1. Since the coordinate system has an **x-axis** and a **y-axis**, lines which intersect the **x-axis** contain the variable **x** in the linear equation; lines which intersect the **y-axis** contain the variable **y** in the linear equation; and, lines which intersect both the **x-axis** and the **y-axis** have both variables **x** and **y** in the linear equation.
2. **Slope-intercept form** of equation of a line is $y = mx + b$ where **m** is the slope of the line and **b** is the **y-intercept** (**y-value** of the point where the line intersects **y-axis**).
3. **Standard form** of the equation of a line is $ax + by = c$ where the number values for **a**, **b**, and **c** are integers (note that the **b** does not represent the **y-intercept** in this form).

GRAPHING

When equation of a line is known, it may be graphed in any of the following ways:

1. **Horizontal lines** have equations which simplify to the form $y = b$, where **b** is the **y-intercept**. The slope of these lines is zero.
2. **Vertical lines** have equations which simplify to form $x = c$, where **c** is the **x-intercept**. They have no slope.
3. **Find at least two points** which make the equation true and are therefore on the line. Finding a third point is one method of checking for errors. If all three points do not form a line then there is an error in at least one of the points. To find these points:
   a. Choose a number at random.
   b. Substitute the number into the linear equation for either the **x** or the **y** variable in the equation.
   c. Solve the resulting equation for the other variables.
   d. The randomly selected number (*step a*) and solution number (*step c*) result in one point: (x, y).
   e. Repeat above *steps a* through *d* above as indicated until the desired number of points have been created.
   f. Plot points and connect them; resulting graph should be a line.
4. **Plot the x-intercept and the y-intercept**.
   a. Substitute zero for the **y** variable in the equation and solve for **x** to find the **x-intercept**.
   b. Substitute zero for the **x** variable in the equation and solve for **y** to find the **y-intercept**.
   c. Plot these two points and draw the graph of the line which contains them.
   d. **NOTE:** Lines which have the same point as the **y-intercept** and the **x-intercept**, that is, the origin **(0,0)**, must have at least one other point located in order to draw the graph of the line.
5. **Write the equation in the slope-intercept form**, plot the point where the line crosses the **y-axis** (the **b** value), use the slope to plot additional points on the line (rise over run). Connect the points to draw the graph of the line.
6. **Find the slope of the line and one point on the line.** Plot the point first, then use the slope to plot additional points on the line. That is, count the slope as rise over run beginning at the point which was just plotted.

LINES (CONTINUED)

FINDING THE EQUATION OF A LINE

- **HORIZONTAL LINES:** the slope is zero and the equation of the line takes the form of $\mathbf{y = b}$, where **b** is the **y-intercept** (the **y**-value of the point of intersection of the line and the **y-axis**).
- **VERTICAL LINES:** there is no slope and the equation of the line takes the form of $\mathbf{x = c}$, where **c** is the **x-intercept** (the **x**-value of the point of intersection of the line and the **x-axis**).
- **NEITHER HORIZONTAL NOR VERTICAL:**
  1. **Given the slope and the y-intercept values**: substitute these numerical values in the slope-intercept form of a linear equation, $\mathbf{y = mx + b}$, where **m** is the slope and **b** is the **y-intercept**.
  2. **Given the slope and one point, either:**
     a. Use the formula for slope, $\mathbf{m = (y_2 - y_1) / (x_2 - x_1)}$, or the **point-slope form** $\mathbf{(x_2 - x_1)\, m = (y_2 - y_1)}$.
        i. Substitute the coordinates from the point for the $\mathbf{x_1}$ and $\mathbf{y_1}$ variables and the slope value for the **m**.
        ii. The equation is then changed to standard form $\mathbf{ax + by = c}$, where **a**, **b**, and **c** are integers.
     b. **Or**, use the slope-intercept form of linear equation, $\mathbf{y = mx + b}$, twice.
        i. The first time substitute the coordinates from the point in the equation for the variables **x** and **y,** and substitute the slope value for the **m**; solve for **b**.
        ii. The second time use the slope-intercept form of a linear equation. Substitute the numerical value for the slope **m** and the intercept **b**, leave the variables **x** and **y** in the equation. The result is the equation of the line in slope-intercept form.
  3. **Given two points:**
     a. Using the points in the slope formula, find the value for the slope, **m.**
     b. Using the slope value and either one of the two points (pick at random), follow the steps given in *item b* above for the slope and one point.
  4. **Given the equation of another line:**
     a. Parallel to the requested line
        i. Use the given equation to find the slope. Parallel lines have the same slope.
        ii. Use this slope value and any other given information and follow the *steps 1, 2, or 3* above, depending on the type of information which is given.
     b. Perpendicular to the requested line
        i. Use the given equation to find the slope. The slope of the requested linear equation is the negative reciprocal of this slope, so change the sign and flip the number to find the slope of the requested line.
        ii. Use this slope value and any other information given in *steps 1, 2, or 3* above, depending on the type of information which is given.

GRAPHING LINEAR INEQUALITIES

- **GRAPHS OF LINEAR INEQUALITIES SUCH AS > AND < ARE HALF PLANES**.
  1. Replace the inequality symbol with = and graph this linear equality as a broken line to indicate that it is only the separation and not part of the graph.
  2. To graph the inequality, randomly pick any point above this line and any point below this line.
  3. Substitute each point into the original inequality.
  4. Whichever point makes the inequality true is in the graph of the inequality so shade all points in the coordinate plane which are on the same side of the line with this point.
- **GRAPHS OF LINEAR INEQUALITIES SUCH AS ≥ AND ≤ INCLUDE BOTH THE HALF PLANES AND THE LINES.**
  1. The same methods given in *item 1* above apply except the line is drawn in solid form because it is part of the graph since the inequalities also include the equal sign.

FINDING THE INTERSECTION OF LINES;
SYSTEMS OF LINEAR EQUATIONS

The purpose of finding the intersection of lines is to find the point which makes two or more equations true at the same time. These equations form a system of equations. These methods are extremely useful in solving word problems.

- **THE SYSTEM OF EQUATIONS IS EITHER:**
  1. **Consistent**; that is, the lines intersect in one point.
  2. **Inconsistent**; that is, the lines are parallel and since they do not intersect, there is no solution to the system of equations. The solution set is the empty set.
  3. **Dependent**; that is, the graphs are the same line. All of the points which make one equation true also make the others true. The lines have all points in common and are therefore dependent equations.

- **TO FIND THE SOLUTION TO A SYSTEM OF EQUATIONS USE ONE OF THESE METHODS:**
  1. **Graph Method** - Graph the equations and locate the point of intersection, if there is one. The point can be checked by substituting the **x** value and the **y** value into all of the equations. If it is the correct point it should make all of the equations true. *This method is weak, since an approximation of the coordinates of the point is often all that is possible.*
  2. **Substitution Method** for solving consistent systems of linear equations includes following steps;
     a. **Solve** one of the equations for one of the variables. It is easiest to solve for a variable which has a coefficient of one (if such a variable coefficient is in the system) because fractions can be avoided until the very end.
     b. **Substitute** the resulting expression for the variable into the **other equation**, not the same equation which was just used.
     c. **Solve** the resulting equation for the remaining variable. This should result in a numerical value for the variable, either **x** or **y**, if the system was originally only two equations.
     d. **Substitute** this numerical value back into one of the original equations and solve for the other variable.
     e. **The solution** is the point containing these **x** and **y** -values, **(x,y)**.
     f. **Check the solution** in all of the original equations.
  3. **Elimination Method** or the **Add/Subtract Method** or the **Linear Combination Method -** eliminate either the **x** or the **y** variable through either addition or subtraction of the two equations. These are the steps for consistent systems of two linear equations;
     a. **Write both equations in the same order**, usually $\mathbf{ax + by = c}$, where **a**, **b**, and **c** are real numbers.
     b. **Observe the coefficients** of the **x** and **y** variables in both equations to determine:
        i. If the **x** coefficients or the **y** coefficients are the same, **subtract** the equations.
        ii. If they are additive inverses (opposite signs: such as **3** and **-3**), **add** the equations.
        iii. If the coefficients of the **x** variables are not the same and are not additive inverses, and the same is true of the coefficients of the **y** variables, then multiply the equations to make one of these conditions true so the equations can be either added or subtracted to eliminate one of the variables.
     c. The above steps should result in **one equation with only one variable, either x or y, but not both**. If the resulting equation has both **x** and **y**, an error was made in following the steps indicated in number *2* above. Correct the error.
     d. **Solve** the resulting equation for the one variable (**x** or **y**).
     e. **Substitute** this numerical value back into either of the original equations and solve for the one remaining variable.
     f. The solution is the point **(x,y)** with the resulting **x** and **y**-values.
     g. **Check** the solution in all of the original equations.
  4. **Matrix method** - involves substantial matrix theory for a system of more than two equations and will not be covered here. Systems of two linear equations can be solved using **Cramer's Rule** which is based on determinants.
     a. For the system of equations: $\mathbf{a_1x + b_1y = c_1}$ and $\mathbf{a_2x + b_2y = c_2}$, where all of the **a**, **b**, and **c** values are real numbers, the point of intersection is $\mathbf{(x,y)}$ where $\mathbf{x = (D_x)/D}$ and $\mathbf{y = (D_y)/D}$.
     b. The determinant **D** in these equations is a numerical value found in this manner:
        $$\mathbf{D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1}$$
     c. The determinant $\mathbf{D_x}$ in these equations is a numerical value found in this manner:
        $$\mathbf{D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1b_2 - c_2b_1}$$
     d. The determinant $\mathbf{D_y}$ in these equations is a numerical value found in this manner:
        $$\mathbf{D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - a_2c_1}$$
     e. Substitute the numerical values found from applying the formulas in *steps b* through *d* into the formulas for **x** and **y** in *step a* above.

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FUNCTIONS

BASIC CONCEPTS

- **RELATION**
  1. Set of ordered pairs; in the coordinate plane, (**x**,**y**).
     a. If a relation, **R**, is the set of ordered pairs (**x**,**y**) then the **inverse** of this relation is the set of ordered pairs (**y**,**x**) and is indicated by the notation $\mathbf{R^{-1}}$.
- **DOMAIN**
  1. Set of the first components of the ordered pairs of the relation; in the coordinate plane, a set of the **x**-values.
- **RANGE**
  1. Set of the second components of the ordered pairs of the relation; in the coordinate plane, a set of the **y**-values.
- **FUNCTION**
  1. Relation in which there is exactly one second component for each of the first components.
     a. **y** is a function of **x** if exactly one value of **y** can be found for each value of **x** in the domain; that is, each **x**-value has only one **y**-value but different **x**-values could have the same **y**-value, so the **y**-values may be used more than once for different **x**-values.
- **VERTICAL LINE TEST**
  1. Indicates a relation is also a function if no vertical line intersects the graph of the relation in more than one point.
- **ONE-TO-ONE FUNCTIONS**
  1. A function, **f**, is one-to-one if $\mathbf{f(a) = f(b)}$ only when $\mathbf{a = b}$
- **HORIZONTAL LINE TEST**
  1. Indicates a one-to-one function if no horizontal line intersects the graph of the function in more than one point.

NOTATION

- **f(x) IS READ AS "f of x"**
  1. Does not indicate the operation of multiplication. Rather, it indicates a function of **x**.
     a. **f(x)** is another way of writing **y** in that the equation $\mathbf{y = x + 5}$ may also be written as $\mathbf{f(x) = x + 5}$ and the ordered pair **(x,y)** may also be written **(x,f(x))**.
     b. To evaluate **f(x)**, use whatever expression is found in the set of parentheses following the **f** to substitute into the rest of the equation for the variable **x**, then simplify completely.
- **COMPOSITE FUNCTIONS: f [g(x)]**
  1. Composition of the function **f** with the function **g**, and it may also be written as $\mathbf{f \circ g(x)}$.
  2. The composition, **f [g(x)]**, is simplified by evaluating the **g** function first and then using this result to evaluate the **f** function.
- **(f + g)(x) EQUALS f(x) + g(x)**
  That is, it represents the sum of the functions.
- **(f - g)(x) EQUALS f(x) - g(x)**
  That is, it represents the difference of the functions.
- **(fg)(x) EQUALS f(x) • g(x)**
  That is, it represents multiplication of the functions.
- **(f/g)(x) EQUALS f(x) / g(x)**
  That is, it represents the division of **f(x)** by **g(x)**.

TYPES OF FUNCTIONS

All linear equations, except those for vertical lines, are functions

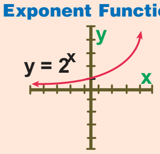
POLYNOMIAL FUNCTIONS

- **WRITTEN FORM**
  1. $\mathbf{f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0}$ for real number values for all of the **a**'s, $\mathbf{a_n \neq 0}$.
- **MAY HAVE TO HAVE RESTRICTED DOMAINS AND/OR RANGES TO QUALIFY AS A FUNCTION**
  1. <u>Without</u> restrictions some equations would only qualify as relations and not functions.
- **FIND THE EQUATION OF THE INVERSE OF A FUNCTION**
  1. Exchange **x** and **y** variables in equation of the function and then solve for **y**. Finally replace **y** with $\mathbf{f^{-1}(x)}$. Not all inverses of functions are also functions.
- **TO GRAPH**
  1. Use the **Remainder Theorem**, if a polynomial **P(x)** is divided by **x - r**, the remainder is **P(r)**, to determine remainders through substitution.
  2. Use the **Factor Theorem**, if a polynomial **P(x)** has a factor **x - r** if and only if $\mathbf{P(r) = 0}$, to find the zeros, roots, and factors of the polynomial.
  3. Find number of **turning points** of graph of a polynomial of degree **n** to be **n – 1** turning points at most.
  4. **Sketch**, using slashed lines, all vertical and/or horizontal asymptotes, if there are any.
  5. **Find the signs** of **P(x)** in intervals between and to each side of the intercepts. This is done to determine the placement of the graph above or below the **x-axis**.
  6. **Plot a few points** in each interval to find the exact graph placement. Also plot all intercepts.
  7. **Note**: The graphs of inverse functions are reflections about the graph of the linear equation $\mathbf{y = x}$.

EXPONENTIAL FUNCTIONS

- **DEFINITION**
  1. An **exponential function** has the form $\mathbf{f(x) = a^n}$, where $\mathbf{a > 0, a \neq 1}$, and the constant real number, **a**, is called the **base**.
- **PROPERTIES**
  1. The graph always intersects the **y-axis** at **(0,1)** because $\mathbf{a^0 = 1}$.
  2. The **domain** is the set of all real numbers.
  3. The **range** is the set of all positive real numbers because **a** is always positive.
  4. When $\mathbf{a > 1}$ the function is increasing; when $\mathbf{a < 1}$ the function is decreasing.
  5. Inverses of exponential functions are **logarithmic functions**.

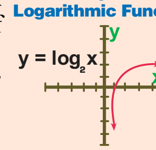
Example of an Exponent Function



LOGARITHMIC FUNCTIONS

- **DEFINITIONS:**
  1. A logarithm is an exponent such that for all positive numbers **a**, where $\mathbf{a \neq 1, y = \log_a x}$ **if and only if** $\mathbf{x = a^y}$; notice that this is the logarithmic function of base **a**.
  2. The **common logarithm**, **log x**, has no base indicated and the understood base is always **10**.
  3. The **natural logarithm**, **ln x**, has no base indicated, is written **ln** instead of **log**, and the understood base is always the number **e**.

Example of a Logarithmic Function



- **PROPERTIES WITH THE VARIABLE a REPRESENTING A POSITIVE REAL NUMBER NOT EQUAL TO ONE:**
  1. $\mathbf{a^{\log_a x} = x}$
  2. $\mathbf{\log_a a^x = x}$
  3. $\mathbf{\log_a a = 1}$
  4. $\mathbf{\log_a 1 = 0}$
  5. If $\mathbf{\log_a u = \log_a v}$, then $\mathbf{u = v}$
  6. If $\mathbf{\log_a u = \log_b u}$ and $\mathbf{u \neq 1}$, then $\mathbf{a = b}$.
  7. $\mathbf{\log_a xy = \log_a x + \log_a y}$
  8. $\mathbf{\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y}$
  9. $\mathbf{\log_a\left(\frac{1}{x}\right) = -\log_a x}$
  10. $\mathbf{\log_a x^n = n(\log_a x)}$, where **n** is a real number.
  11. Change of Base Rule: If $\mathbf{a > 0, a \neq 1, b > 0, b \neq 1}$, and $\mathbf{x > 0}$ then $\mathbf{\log_a x = \frac{(\log_b x)}{(\log_b a)}}$
  12. Finding Natural Logarithms: $\mathbf{\ln x = \frac{(\log x)}{(\log a)}}$
- **COMMON MISTAKES!**
  1. $\mathbf{\log_a (x+y) = \log_a x + \log_a y}$ FALSE!
  2. $\mathbf{\log_a x^n = (\log_a x)^n}$ FALSE!
  3. $\mathbf{\frac{(\log_a x)}{(\log_a y)} = \log_a (x - y)}$ FALSE!
- **SOLVING LOGARITHMIC EQUATIONS**
  1. Put all logarithm expressions on one side of the equals sign.
  2. Use the properties to simplify the equation to one logarithm statement on one side of the equals sign.
  3. Convert the equation to the equivalent exponential form.
  4. Solve and check the solution.

RATIONAL FUNCTIONS

Definition: $f(x)=\frac{P(x)}{Q(x)}$ *where* $P(x)$ *and* $Q(x)$ *are polynomials which are relatively prime (lowest terms),* $Q(x)$ *has degree greater than zero, and* $Q(x) \neq 0$.

TO GRAPH

- **DOMAIN**
  1. The domain is all real numbers except for those numbers which make $Q(x) = 0$.
- **INTERCEPTS**
  1. **y-intercept:** Set $x = 0$ and solve for y. There is one y-intercept. If $Q(x) = 0$ when $x = 0$ then y is undefined and the function does not intersect the y-axis.
  2. **x-intercepts**: Set $y = 0$. Since $f(x)=\frac{P(x)}{Q(x)}$ can equal zero only when $P(x) = 0$, the **x-**intercepts are the roots of the equation $P(x) = 0$.

ASYMPTOTES

A line which the graph of the function approaches, getting closer with each point, but never intersects.

- **HORIZONTAL ASYMPTOTES**
  1. Horizontal asymptotes exist when the degree of $P(x)$ is less than or equal to the degree of $Q(x)$.
  2. The **x-**axis is a horizontal asymptote whenever $P(x)$ is a constant and has degree equal to zero.
  3. Steps to find horizontal asymptotes
     a. Factor out the highest power of x found in $P(x)$.
     b. Factor out the highest power of x found in $Q(x)$.
     c. Reduce the function; that is, cancel common factors found in $P(x)$ and $Q(x)$.
     d. Let $|x|$ increase, and disregard all fractions in $P(x)$ and in $Q(x)$ which have any power of x greater than zero in the denominators; because these fractions approach zero and may be disregarded completely.
     e. **When the result of the previous step is:**
        i. a constant, c, the equation of the horizontal asymptote is $y = c$.
        ii. a fraction such as c/x^n where c is a constant and $n \neq 0$, the asymptote is the **x-axis**.
        iii. neither a constant nor a fraction, there is no horizontal asymptote.
- **VERTICAL ASYMPTOTES**
  1. Vertical asymptotes exist for values of x which make $Q(x) = 0$; that is, for values of x which make the denominator equal to zero, and therefore make the rational expression undefined.
  2. There can be several vertical asymptotes.
  3. Steps to find vertical asymptotes:
     a. Set the denominator, $Q(x)$, equal to zero
     b. Factor if possible
     c. Solve for x
     d. The vertical asymptotes are vertical lines whose equations are of the form $x = r$, where r is a solution of $Q(x) = 0$ because each r value will make the denominator, $Q(x)$, equal to zero when it is substituted for x into $Q(x)$.

SYMMETRY

- **DESCRIPTION:**
  1. Graphs are symmetric with respect to a line if, when folded along the drawn line, and the two parts of the graph then land upon each other.
  2. Graphs are symmetric with respect to the origin if, when the paper is folded twice, the first fold being along the **x-axis** (do not open this fold before completing the second fold) and the second fold being along the **y-axis**, the two parts of the graph land upon each other.
- **GRAPHS ARE SYMMETRIC WITH RESPECT TO:**
  1. The **x-axis** if replacing y with $-y$ results in an equivalent equation;
  2. The **y-axis** if replacing x with $-x$ results in an equivalent equation;
  3. The **origin** if replacing both x with $-x$ and y with $-y$ results in an equivalent equation.
- **DETERMINE POINTS**
  1. Create a few points, by substituting values for x and solving for $f(x)$, which make the rational function equation true.
  2. Include points from each region created by the vertical asymptotes (choose values for x from these regions).
  3. Include the **y-intercept** (if there is one) and any **x-intercept**s.
  4. Apply symmetry (if the graph is found to be symmetric after testing for symmetry) to find additional points; that is, if the graph is symmetric with respect to the **x-axis** and point **(3,-7)** makes the equation $f(x)$ true, then the point **(-3,-7)** will be on the graph and should also make the equation true.
- **PLOT THE GRAPH**
  1. Sketch any horizontal or vertical asymptotes by drawing them as broken or dashed lines.
  2. Plot the points, some from each region created by the vertical asymptotes, which make the equation $f(x)$ true.
  3. Draw the graph of the rational function equation, $f(x) = P(x) / Q(x)$, applying any symmetry which applies.

SEQUENCES AND SERIES

DEFINITIONS

- **INFINITE SEQUENCE:** is a function with a domain which is the set of positive integers; written as $a_1, a_2, a_3, \ldots$ with each a_i representing a term.
- **FINITE SEQUENCE:** is a function with a domain of only the first n positive integers; written as $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$
- **SUMMATION**: $\sum_{k=1}^{m} a_k = a_1 + a_2 + \ldots + a_{m-1} + a_m$ where k is the index of the summation and is always an integer which begins with the value found at the bottom of the summation sign and increases by 1 until it ends with the value written at the top of the summation sign.
- ***n*TH PARTIAL SUM:** $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_{n-1} + a_n$
- **ARITHMETIC SEQUENCE OR ARITHMETIC PROGRESSION:** is a sequence in which each term differs from the preceding term by a constant amount, called the common difference; that is, $a_n = a_{n-1} + d$ where d is the common difference.
- **GEOMETRIC SEQUENCE OR GEOMETRIC PROGRESSION:** is a sequence in which each term is a constant multiple of the preceding term; that is, $a_n = ra_{n-1}$ where r is the constant multiple and is called the common ratio.
- $n! = n(n-1)(n-2)(n-3)\ldots(3)(2)(1)$; this is read "**n factorial**." NOTE: $0! = 1$

PROPERTIES OF SUMS, SEQUENCES AND SERIES

1. $\sum_{k=1}^{n}(a_k = b_k) = \sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k$
2. $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$, where c is a constant
3. $\sum_{k=1}^{n} c = nc$, where c is a constant
4. The ***n*th** term of an arithmetic sequence is $a_n = a_1 + (n-1)d$, where d is common difference.
5. The sum of the first n terms of an arithmetic sequence with a_1 as the first term and d as the common difference is
$$s_n = \frac{n}{2}(a_1 + a_n) \text{ or } s_n = \frac{n}{2}[2a_1 + (n-1)d]$$
6. The *n*th term of a geometric sequence with a_1 as the first term and r as the common ratio is $a_n = a_1 r^{n-1}$.
7. The sum of the first n terms of a geometric sequence with a_1 as the first term and r as the common ratio and $r \neq 1$ is $S_n = \frac{[a_1(r^n - 1)]}{(r-1)}$
8. The sum of the terms of an infinite geometric sequence with a_1 as the first term and r as the common ratio where $|r| < 1$ is $\frac{a_1}{1-r}$ if $|r| > 1$ or $|r| = 1$, the sum does not exist.
9. The rth term of the binomial expansion of $(x+y)^n$ is
$$\frac{n!}{[n-(r-1)]!(r-1)!}x^{n-(r-1)}y^{(r-1)}$$

CONIC SECTIONS

*The charts below contain all general equation forms of conic sections. These general forms can be used both to graph and to determine equations of conic sections. The values for **h** and **k** can be any real number, including zero.*

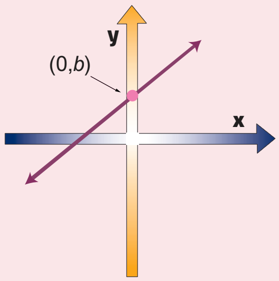
DESCRIPTION

Conic sections represent the intersections of a plane and a right circular cone; that is, parabolas, circles, ellipses, and hyperbolas. In addition, when the plane passes through the vertex of the cone it may determine a degenerate conic section; that is, a point, line, or two intersecting lines.

GENERAL EQUATION

The general form of the equation of a conic section with axes parallel to the coordinate axes is:

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ where **A** and **C** are not both zero.

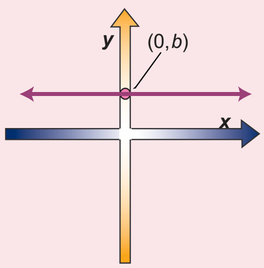


TYPE: LINE

GENERAL EQUATION: $y = mx + b$

Notation:
1. **m** is slope
2. **b** is **y-intercept**

Values:
1. $m > 0$ then the line is higher on the right end.
2. $m < 0$ then the line is higher on the left end

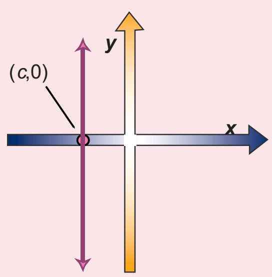


TYPE: HORIZONTAL LINE

GENERAL EQUATION: $y = b$

Notation: **b** is **y-intercept**

Values: $m = 0$ then the line is horizontal through $(0, b)$

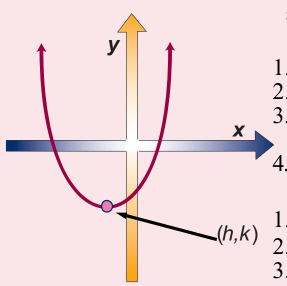


TYPE: VERTICAL LINE

GENERAL EQUATION: $x = c$

Notation: **c** is **x-intercept**

Values:
1. no slope
2. vertical line through $(c, 0)$



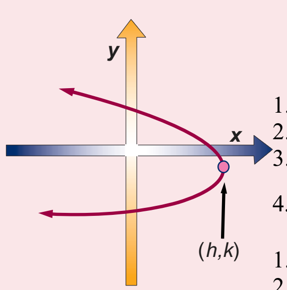
TYPE: PARABOLA

GENERAL EQUATION: $y = a(x - h)^2 + k$

STANDARD FORM: $(x - h)^2 = 4p(y - k)$

Notation:
1. x^2 term and y^1 term
2. (h,k) is vertex
3. $(h, k \pm p)$ is center of focus where $p = \frac{1}{4a}$
4. $y = k \pm p$ is directrix equation where $p = \frac{1}{4a}$

Value:
1. $a > 0$ then opens up.
2. $a < 0$ then opens down
3. $x = h$ is equation of line of symmetry
4. larger $|a|$ = thinner parabola; smaller $|a|$ = fatter parabola



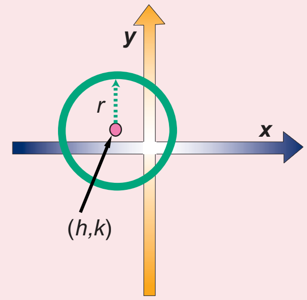
TYPE: PARABOLA

GENERAL EQUATION: $x = a(y - k)^2 + h$

STANDARD FORM: $(y - k)^2 = 4p(x - h)$

Notation:
1. x^1 term and y^2 term
2. (h, k) is vertex
3. $(h \pm p, k)$ is focus where $p = \frac{1}{4a}$
4. $x = h \pm p$ is directrix equation where $p = \frac{1}{4a}$

Values:
1. $a > 0$ then opens right.
2. $a < 0$ then opens left
3. $y = k$ is equation of line of symmetry



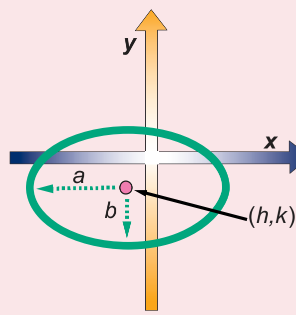
TYPE: CIRCLE

GENERAL EQUATION:

$(x - h)^2 + (y - k)^2 = r^2$

Notation:
1. x^2 term and y^2 term both with the same positive coefficient
2. r^2 is a positive number
3. (h, k) is center
4. **r** is radius

Values: none



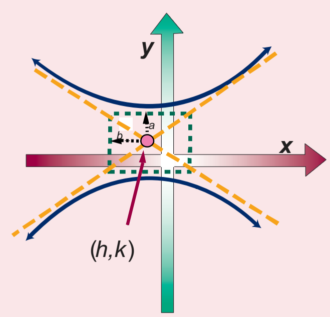
TYPE: ELLIPSE

GENERAL EQUATION:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2}$$

Notation:
1. x^2 term and y^2 term with different coefficients
2. (h, k) is center
3. **a** is horizontal distance to left and right of (h, k)
4. **b** is vertical distance above and below (h, k)

Values:
1. $a > b$ then major axis is horizontal and foci are $(h \pm c, k)$ where $c^2 = a^2 - b^2$
2. $b > a$ then major axis is vertical and foci are $h, k \pm c)$ where $c^2 = b^2 - a^2$



TYPE: HYPERBOLA

GENERAL EQUATION:

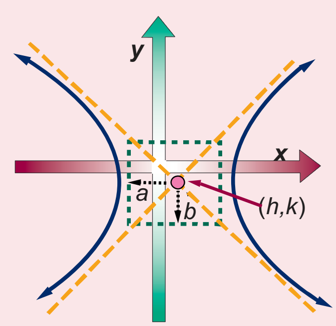
$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Notation:
1. x^2 term and y^2 term with a negative coefficient for x^2 term
2. (h, k) is center of a rectangle
3. **b** is horizontal distance to left and right of (h, k)
4. **a** is vertical distance above and below (h, k) to the vertices

Values:

$y = \frac{a}{b}x$ and $y = -\frac{a}{b}x$

are equations of asymptotes



TYPE: HYPERBOLA

GENERAL EQUATION:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Notation:
1. x^2 term and y^2 term with a negative coefficient for y^2 term
2. (h, k) is center of a rectangle
3. **a** is horizontal distance to left and right of (h, k) to the vertices
4. **b** is vertical distance above and below (h, k)

Values:

$y = \frac{a}{b}x$ and $y = -\frac{a}{b}x$

are equations of asymptotes

PROBLEM SOLVING

DIRECTIONS

1. Read the problem carefully.
2. Note the given information, the question asked and the value requested.
3. Categorize the given information, removing unnecessary information.
4. Read the problem again to check for accuracy, to determine what, if any, formulas are needed and to establish the needed variables.
5. Write the needed equation(s) and determine the method of solution to use; this will depend on the degree of the equations, the number of variables and the number of equations.
6. Solve the problem. Check the solution. Read the problem again to make sure the answer given is the one requested.

ODD NUMBERS, EVEN NUMBERS, MULTIPLES

NOTATION

d is the common difference between any two consecutive numbers of a set of numbers.

FORMULAS

First number = **x** Second number = **x+d**
Third number = **x+2d** Fourth number = **x+3d**; etc.

Example: The first 5 multiples of 3 are **x**, **x+3**, **x+6**, **x+9**, and **x+12** because **d = 3**

RECTANGLES

NOTATION

P is perimeter; **l** is length; **w** is width; **A** is area

FORMULAS

1. $P = 2l + 2w$ 2. $A = lw$

Example: The length of a rectangle is **5** more than the width and the perimeter is **38**.
Equation: $38 = 2(w + 5) + 2w$

TRIANGLES

NOTATION

P is perimeter; **S** is side length; **A** is area; **a** is altitude; **b** is base
NOTE: *altitude and base must be perpendicular i.e. form 90° angles.*

FORMULAS

1. $P = S_1 + S_2 + S_3$ 2. $A = {}^1/_2\, ab$

Example: The base of a triangle is 3 times the altitude and the area is 24.
Equation: $24 = {}^1/_2 \bullet a \bullet 3a$

CIRCLE

NOTATION

C is circumference; **A** is area; **d** is diameter; **r** is radius; π is pi = 3.14...

FORMULAS

1. $C = \pi d$ 2. $A = \pi r^2$ 3. $d = 2r$

Example: The radius of a circle is 4 and the circumference is 25.12.
Equation: $25.12 = \pi \bullet 8$

PYTHAGOREAN THEOREM

NOTATION

a is a leg; **b** is a leg; **c** is a hypotenuse
NOTE: *Hypotenuse is the longest side*

FORMULA

$$a^2 + b^2 = c^2$$

NOTE: *Applies to right triangles only*.

Example: The hypotenuse of a right triangle is 2 times the shortest leg. The other leg is $\sqrt{3}$ times the shortest leg.
Equation: $a^2 + (\sqrt{3}\,a)^2 = (2a)^2$

MONEY, COINS, BILLS, PURCHASES

NOTATION

V is currency value; **C** is number of coins, bills, or purchased items

FORMULA

$$V_1C_1 + V_2C_2 = V_{total}$$

Example: Jack bought black pens at $1.25 each and blue pens at $0.90 each. He bought 5 more blue pens than black pens and spent $36.75.
Equation: $1.25x + 0.90(x+5) = 36.75$

MIXTURE

NOTATION

V_1 is first volume; P_1 is first percent solution; V_2 is second volume; P_2 is second percent solution; V_F is final volume; P_F is final percent solution
NOTE: Water could be 0% solution and pure solution could be 100%.

FORMULA

$$V_1P_1 + V_2P_2 = V_FP_F$$

Example: How much water should be added to 20 liters of 80% acid solution to yield 70% acid solution?
Equation: $x(0) + 20(0.80) = (x+20)(0.70)$

WORK

NOTATION

W_1 is rate of one person or machine multiplied by the time it would take for the entire job to be completed by 2 or more people or machines;
W_2 is the rate of the second person or machine multiplied by time for entire job;
1 represents the whole job.
NOTE: *Rate is the part of the job completed by one person or machine.*

FORMULA

$$W_1 + W_2 = 1$$

Example: John can paint a house in 4 days, while Sam takes 5 days. How long would they take if they worked together?
Equation: $\frac{1}{4}x + \frac{1}{5}x = 1$

DISTANCE

NOTATION

d is distance; **r** is rate; i.e. speed; **t** is time; value indicated in the speed, i.e. miles per hour has time in hours
NOTE: *Add or subtract speed of wind or water current with the rate;* ***(r ± wind)*** *or* ***(r ± current).***

FORMULAS

1. $d = rt$
   Example: John traveled 200 miles in 4 hours.
   Equation: $200 = r \cdot 4$
2. $d_{to} = d_{returning}$
   Example: With a 30 mph head wind a plane can fly a certain distance in 6 hours. Returning, flying in opposite direction, it takes one hour less.
   Equation: $(r - 30)6 = (r + 30)5$
3. $d_1 + d_2 = d_{total}$
   Example: Lucy and Carol live 400 miles apart. They agree to meet at a shopping mall located between their homes. Lucy drove at 60mph, and Carol drove at 50mph and left one hour later.
   Equation: $60t + 50(t-1) = 400$

SIMPLE INTEREST

NOTATION

I is interest; **P** is principal, amount borrowed, saved, or loaned;
S is total amount, or **I + P**; **r** is % interest rate;
t is time expressed in years; **p** is monthly payment

FORMULAS

1. $I = Prt$
   Example: Anna borrowed $800 for 2 years and paid $120 interest.
   Equation: $120 = 800\, r(2)$
2. $S = P + Prt$
   Example: Alex borrowed $4600 at 9.3% for 6 months.
   Equation: $S = 4600 + 4600\,(.093)(.5)$
   NOTE**: 9.3% = .093** and **6 months = 0.5 year**
3. $p = \dfrac{P + Prt}{t\ 12}$
   Example: Evan borrowed $3,000 for a used car and is paying it off monthly over 2 years at 10% interest.
   Equation: $p = [3{,}000 + 3000\,(.1)(2)] / (2)(12)$

PROPORTION AND VARIATION

NOTATION

a, b, c, d, are quantities specified in the problem; $k \neq 0$.

FORMULAS

1. **Proportion:** $\frac{a}{b} = \frac{c}{d}$; cross multiply to get $ad = bc$
2. **Direct Variation:** $y = kx$
3. **Inverse variation:** $y = \frac{k}{x}$

Examples:

1. **Proportion**: If 360 acres are divided between John and Bobbie in the ratio 4 to 5, how many acres does each receive?
   Equation: $\frac{John}{Bobbie}$ so $\frac{4}{5} = \frac{J}{360 - J}$
2. **Direct Variation:** If the price of gold varies directly as the square of its mass, and 4.2 grams of gold is worth $88.20, what will be the value of 10 grams of gold?
   Equation: $88.20 = k(4.2)^2$; solve to find $k = 5$; then use the equation $y = 5(10)^2$ where **y** is the value of **10** grams of gold.
3. **Inverse Variation:** If **a** varies inversely as **b** and, and **a = 4** when **b = 10**, find **a** when **b = 5**.
   Equation: $4 = \frac{k}{10}$ so $k = 40$; then $a = \frac{40}{5}$ to find **a**.

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