Άσκηση 1 (Ίδιο με Θέμα 2ο → Ιούνιος 2017)

Θεωρούμε ως τυχαία μεταβλητή $f(x_i;\lambda_i) = \dfrac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!}, \ x_i = 0,1,2,\dots$

$X_1, X_2, X_3, \dots, X_n$ με $X_i \sim \text{Poisson}(\lambda)$

$E(X_i) = Var(X_i) = \lambda_i$

Από γραμμικό μοντέλο $\rightarrow X_i = \lambda + u_i$

(i) $$\prod_{i=1}^{n} f(x_i,\lambda) = f(x_1;\lambda)\cdot f(x_2;\lambda)\cdots f(x_n;\lambda)$$

$$= \frac{e^{-\lambda}\lambda^{x_1}}{x_1!}\cdot\frac{e^{-\lambda}\lambda^{x_2}}{x_2!}\cdots\frac{e^{-\lambda}\lambda^{x_n}}{x_n!} = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n}x_i}}{\prod_{i=1}^{n}x_i!} = L(\lambda;x_i)$$

$$\ln L = \ln\left[\frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n}x_i}}{\prod_{i=1}^{n}x_i!}\right] = \ln\left[e^{-n\lambda}\lambda^{\sum_{i=1}^{n}x_i}\right] - \ln\left(\prod_{i=1}^{n}x_i!\right)$$

$$= \ln(e^{-n\lambda}) + \ln\left(\lambda^{\sum_{i=1}^{n}x_i}\right) - \ln\left(\prod_{i=1}^{n}x_i!\right)$$

$$= -n\lambda + \sum_{i=1}^{n}x_i\ln\lambda - \ln\left(\prod_{i=1}^{n}x_i!\right)$$

F.O.C: $$\frac{\partial \ln L}{\partial\lambda} = -n + \frac{1}{\lambda}\sum_{i=1}^{n}x_i + 0 = 0$$

$$\Leftrightarrow \frac{1}{\lambda}\sum_{i=1}^{n}x_i = n \Leftrightarrow \boxed{\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n}x_i}$$

S.O.C: $$\frac{\partial^2 \ln L}{\partial\lambda^2} = 0 - \frac{1}{\lambda^2}\overbrace{\sum_{i=1}^{n}x_i}^{=n\lambda} = -\frac{1}{\lambda^2}n\lambda = -\frac{n}{\lambda}$$

άρα μέγιστο

(ii) Αμεροληψία

- $E(\hat{\lambda}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) =$

$$= \frac{1}{n}\sum_{i=1}^{n}\lambda = \frac{1}{\cancel{n}}\cdot \cancel{n}\lambda = \lambda$$

Συνέπεια

$$Var(\hat{\lambda}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) =$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\lambda = \frac{1}{n^{\cancel{2}}}\cdot \cancel{n}\lambda = \frac{\lambda}{n}$$

Ικανές συνθήκες

- $\lim_{n\to+\infty} E(\hat{\lambda}) = \lim_{n\to+\infty}\lambda = \lambda$ ✓
- $\lim_{n\to+\infty} Var(\hat{\lambda}) = \lim_{n\to+\infty}\frac{\lambda}{n} = 0$ ✓

Άρα η εκτιμήτρια είναι συνεπής.

Πλήρη αποτελεσματικότητα

$$C\text{-}R = I_n^{-1}(\hat{\lambda}) = \left[E\left(-\frac{\partial^2 \ln L}{\partial \lambda^2}\right)\right]^{-1} = \left[E\left[-\left(-\frac{n}{\lambda}\right)\right]\right]^{-1}$$

$$= \frac{\lambda}{n} = Var(\hat{\lambda})$$

Επομένως η εκτιμήτρια είναι πλήρως αποτελεσματική.

Άσκηση 2 (Ίδιο θέμα 2ο → Ιούλιος 2017)

Τυχαίο δείγμα $X_1, X_2, \dots, X_n$ με $X_i \sim Bernoulli(\theta)$

$E(X_i) = \theta$, $Var(X_i) = \theta(1-\theta)$

$$f(x_i;\theta) = \theta^{x_i}\cdot(1-\theta)^{1-x_i}, \quad 0 \le \theta \le 1, \; X_i \in \{0,1\}$$

Απλό γραμμικό μοντέλο → $X_i = \theta + u_i$

(i) $$\prod_{i=1} f(x_i;\theta) = f(x_1;\theta)\, f(x_2;\theta) \dots f(x_n;\theta)$$

$$= \theta^{x_1}(1-\theta)^{1-x_1}\cdot\theta^{x_2}(1-\theta)^{1-x_2}\dots\theta^{x_n}(1-\theta)^{1-x_n}$$

$$= \theta^{\sum_{i=1}^{n}x_i}(1-\theta)^{\sum_{i=1}^{n}(1-x_i)} = \theta^{\sum_{i=1}^{n}x_i}\cdot(1-\theta)^{\sum_{i=1}^{n}1-\sum_{i=1}^{n}x_i}$$

$$= \theta^{\sum_{i=1}^{n}x_i}\cdot(1-\theta)^{n-\sum_{i=1}^{n}x_i} = L(\theta; x_i)$$

$$\ln L = \ln\left[\theta^{\sum_{i=1}^{n}x_i}\cdot(1-\theta)^{n-\sum_{i=1}^{n}x_i}\right] =$$

$$= \ln\left[\theta^{\sum_{i=1}^{n}x_i}\right] + \ln\left[(1-\theta)^{n-\sum_{i=1}^{n}x_i}\right] =$$

$$= \sum_{i=1}^{n}x_i\cdot\ln\theta + \left(n - \sum_{i=1}^{n}x_i\right)\cdot\ln(1-\theta)$$

F.O.C.: $$\frac{\partial \ln L}{\partial\theta} = \frac{1}{\theta}\sum_{i=1}^{n}x_i - \frac{1}{1-\theta}\left[n - \sum_{i=1}^{n}x_i\right] = 0$$

$$\xrightarrow{\theta(1-\theta)} (1-\theta)\sum_{i=1}^{n}x_i - \theta\left[n - \sum_{i=1}^{n}x_i\right] = 0$$

$$\Leftrightarrow \sum_{i=1}^{n}x_i - \cancel{\theta\sum_{i=1}^{n}x_i} - \theta n + \cancel{\theta\sum_{i=1}^{n}x_i} = 0 \Leftrightarrow \boxed{\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}x_i}$$

$$\underline{S.O.C:} \quad \frac{\partial^2 \ln L}{\partial \theta^2} = \frac{-1}{\theta^2} \sum_{i=1}^{n} x_i - \frac{1}{(1-\theta)^2}\left[n - \sum_{i=1}^{n} x_i\right] \qquad (4)$$

$$= -\frac{1}{\theta^{\cancel{2}}} n \cancel{\theta} - \frac{1}{(1-\theta)^2}\left(n - n\theta\right) =$$

$$= -\frac{n}{\theta} - \frac{1}{(1-\theta)^{\cancel{2}}} n \cancel{(1-\theta)} = -\frac{n}{\theta} - \frac{n}{1-\theta}$$

$$= \frac{-n(1-\theta) - n\theta}{\theta(1-\theta)} = \frac{-n + \cancel{n\theta} - \cancel{n\theta}}{\theta(1-\theta)} = \frac{-n}{\theta(1-\theta)} < 0$$

άρα μέγιστο. αφού $\theta \in [0,1]$

(ii) <u>Αμεροληψία</u>

- $E(\hat{\theta}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) =$

$$= \frac{1}{n}\sum_{i=1}^{n} \theta = \frac{1}{n} n\theta = \theta$$

Άρα αμερόληπτος

<u>Συνέπεια</u>

$$Var(\hat{\theta}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} x_i$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(x_i) = \frac{1}{n^2}\sum_{i=1}^{n} \theta(1-\theta)$$

$$= \frac{1}{n^2} n\theta(1-\theta) = \frac{\theta(1-\theta)}{n}$$

Ικανές Συνθήκες

$$\lim_{n\to\infty} E(\hat{\theta}) = \lim_{n\to\infty} \theta = \theta \;\checkmark$$

$$\lim_{n\to\infty} Var(\hat{\theta}) = \lim_{n\to\infty} \left[\frac{\theta(1-\theta)}{n}\right] = 0 \;\checkmark$$

Άρα η εκτιμήτρια είναι συνεπής.

Πλήρης αποτελεσματικότητα

$$C\text{-}R = I_n^{-1}(\theta) = \left[E\left(-\frac{\partial^2 \ln L}{\partial \theta^2}\right)\right]^{-1} =$$

$$= \left[E\left[-\left(-\frac{n}{\theta(1-\theta)}\right)\right]\right]^{-1} =$$

$$= \left[E\left(\frac{n}{\theta(1-\theta)}\right)\right]^{-1} = \frac{\theta(1-\theta)}{n} = Var(\hat{\theta})$$

Άρα ο εκτιμητής είναι πλήρως αποτελεσματικός