DIPLOMA PROGRAMME

MATHEMATICS HL
FURTHER MATHEMATICS SL
INFORMATION BOOKLET

For use by teachers and students, during the course and in the examinations

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Formulae

Presumed knowledge

Area of a parallelogram	$A = (b \times h)$, where b is the base, h is the height
Area of a triangle	$A = \frac{1}{2}(b \times h)$, where b is the base, h is the height
Area of a trapezium	$A = \frac{1}{2}(a + b)h$, where a and b are the parallel sides, h is the height
Area of a circle	$A = \pi r^2$, where r is the radius
Circumference of a circle	$C = 2\pi r$, where r is the radius
Volume of a pyramid	$V = \frac{1}{3}$(area of base $\times$ vertical height)
Volume of a cuboid	$V = l \times w \times h$, where l is the length, w is the width, h is the height
Volume of a cylinder	$V = \pi r^2 h$, where r is the radius, h is the height
Area of the curved surface of a cylinder	$A = 2\pi rh$, where r is the radius, h is the height
Volume of a sphere	$V = \frac{4}{3}\pi r^3$, where r is the radius
Volume of a cone	$V = \frac{1}{3}\pi r^2 h$, where r is the radius, h is the height
Distance between two points (x_1, y_1) and (x_2, y_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
Coordinates of the midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

Topic 1—Core: Algebra

1.1	The n^{th} term of an arithmetic sequence	$u_n = u_1 + (n-1)d$
	The sum of n terms of an arithmetic sequence	$S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$
	The n^{th} term of a geometric sequence	$u_n = u_1 r^{n-1}$
	The sum of n terms of a finite geometric sequence	$S_n = \frac{u_1(r^n - 1)}{r-1} = \frac{u_1(1-r^n)}{1-r}$, $r \neq 1$
	The sum of an infinite geometric sequence	$S = \frac{u_1}{1-r}$, $\lvert r \rvert < 1$
1.2	Exponents and logarithms	$a^x = b \Leftrightarrow x = \log_a b$ $a^x = \mathrm{e}^{x \ln a}$ $\log_a a^x = x = a^{\log_a x}$ $\log_b a = \frac{\log_c a}{\log_c b}$
1.3	Combinations	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$
	Binomial theorem	$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$
1.5	Complex numbers	$z = a + \mathrm{i}b = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{i\theta} = r\,\mathrm{cis}\,\theta$
1.7	De Moivre's theorem	$[r(\cos\theta + \mathrm{i}\sin\theta)]^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta) = r^n\mathrm{e}^{in\theta} = r^n\,\mathrm{cis}\,n\theta$

Topic 2—Core: Functions and equations

2.5	Axis of symmetry of the graph of a quadratic function	$f(x) = ax^2 + bx + c \Rightarrow$ axis of symmetry $x = -\frac{b}{2a}$
2.6	Solution of a quadratic equation	$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $a \neq 0$
	Discriminant	$\Delta = b^2 - 4ac$

Topic 3—Core: Circular functions and trigonometry

3.1	Length of an arc	$l = \theta r$, where θ is the angle measured in radians, r is the radius
	Area of a sector	$A = \frac{1}{2}\theta r^2$, where θ is the angle measured in radians, r is the radius
3.2	Identities	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
	Pythagorean identities	$\cos^2\theta + \sin^2\theta = 1$ $1 + \tan^2\theta = \sec^2\theta$ $1 + \cot^2\theta = \csc^2\theta$
3.3	Compound angle identities	$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$ $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$ $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
	Double angle identities	$\sin 2\theta = 2\sin\theta\cos\theta$ $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$ $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$
3.6	Cosine rule	$c^2 = a^2 + b^2 - 2ab\cos C$; $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$
	Sine rule	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
	Area of a triangle	$A = \frac{1}{2}ab\sin C$, where a and b are adjacent sides, C is the included angle

Topic 4—Core: Matrices

4.3	Determinant of a 2×2 matrix	$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det \boldsymbol{A} = ad - bc$
	Inverse of a 2×2 matrix	$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \boldsymbol{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $ad \neq bc$
	Determinant of a 3×3 matrix	$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \Rightarrow \det \boldsymbol{A} = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Topic 5—Core: Vectors

5.1	Magnitude of a vector	$\lvert \boldsymbol{v} \rvert = \sqrt{v_1^{\,2} + v_2^{\,2} + v_3^{\,2}}$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
	Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
	Coordinates of the midpoint of a line segment with endpoints (x_1, y_1, z_1), (x_2, y_2, z_2)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$
5.2	Scalar product	$\boldsymbol{v} \cdot \boldsymbol{w} = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \cos\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
	Angle between two vectors	$\cos\theta = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{\lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert}$
5.3	Vector equation of a line	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$
	Parametric form of equations of a line	$x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$
	Cartesian equations of a line	$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$
5.5	Vector product (Determinant representation)	$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ $\lvert \boldsymbol{v} \times \boldsymbol{w} \rvert = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \sin\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$
	Area of a triangle	$A = \frac{1}{2} \lvert \boldsymbol{v} \times \boldsymbol{w} \rvert$
5.6	Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$
	Equation of a plane (using the normal vector)	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
	Cartesian equation of a plane	$ax + by + cz + d = 0$

Topic 6—Core: Statistics and probability

6.3		Let $n = \sum_{i=1}^{k} f_i$
	Population parameters	
	Mean μ	$\mu = \frac{\sum_{i=1}^{k} f_i x_i}{n}$
	Variance σ^2	$\sigma^2 = \frac{\sum_{i=1}^{k} f_i (x_i - \mu)^2}{n} = \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2$
	Standard deviation σ	$\sigma = \sqrt{\frac{\sum_{i=1}^{k} f_i (x_i - \mu)^2}{n}}$
	Sample statistics	
	Mean $\bar{x}$	$\bar{x} = \frac{\sum_{i=1}^{k} f_i x_i}{n}$
	Variance s_n^2	$s_n^2 = \frac{\sum_{i=1}^{k} f_i (x_i - \bar{x})^2}{n} = \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - \bar{x}^2$
	Standard deviation s_n	$s_n = \sqrt{\frac{\sum_{i=1}^{k} f_i (x_i - \bar{x})^2}{n}}$
	Unbiased estimate of population variance s_{n-1}^2	$s_{n-1}^2 = \frac{n}{n-1} s_n^2 = \frac{\sum_{i=1}^{k} f_i (x_i - \bar{x})^2}{n-1} = \frac{\sum_{i=1}^{k} f_i x_i^2}{n-1} - \frac{n}{n-1}\bar{x}^2$
6.5	Probability of an event A	$\mathrm{P}(A) = \frac{n(A)}{n(U)}$
	Complementary events	$\mathrm{P}(A) + \mathrm{P}(A') = 1$
6.6	Combined events	$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$
	Mutually exclusive events	$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$

Topic 6—Core: Statistics and probability (continued)

6.7	Conditional probability	$\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$
	Independent events	$\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$
	Bayes' Theorem	$\mathrm{P}(B \mid A) = \dfrac{\mathrm{P}(B)\mathrm{P}(A \mid B)}{\mathrm{P}(B)\mathrm{P}(A \mid B) + \mathrm{P}(B')\mathrm{P}(A \mid B')}$
6.9	Expected value of a discrete random variable X	$\mathrm{E}(X) = \mu = \sum_x x\,\mathrm{P}(X = x)$
	Expected value of a continuous random variable X	$\mathrm{E}(X) = \mu = \int_{-\infty}^{\infty} x\,f(x)\,\mathrm{d}x$
	Variance	$\mathrm{Var}(X) = \mathrm{E}(X-\mu)^2 = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$
	Variance of a discrete random variable X	$\mathrm{Var}(X) = \sum (x-\mu)^2\,\mathrm{P}(X = x) = \sum x^2\,\mathrm{P}(X = x) - \mu^2$
	Variance of a continuous random variable X	$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x - \mu^2$
6.10	Binomial distribution	$X \sim \mathrm{B}(n, p) \Rightarrow \mathrm{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x},\ x = 0, 1, \ldots, n$
	Mean	$\mathrm{E}(X) = np$
	Variance	$\mathrm{Var}(X) = np(1-p)$
	Poisson distribution	$X \sim \mathrm{P_o}(m) \Rightarrow \mathrm{P}(X = x) = \dfrac{m^x \mathrm{e}^{-m}}{x!},\ x = 0, 1, 2, \ldots$
	Mean	$\mathrm{E}(X) = m$
	Variance	$\mathrm{Var}(X) = m$
6.11	Standardized normal variable	$z = \dfrac{x-\mu}{\sigma}$

Topic 7—Core: Calculus

7.1	Derivative of $f(x)$	$y = f(x) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = \lim_{h \to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$
	Derivative of x^n	$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$
	Derivative of $\sin x$	$f(x) = \sin x \Rightarrow f'(x) = \cos x$
	Derivative of $\cos x$	$f(x) = \cos x \Rightarrow f'(x) = -\sin x$
	Derivative of $\tan x$	$f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$
	Derivative of e^x	$f(x) = \mathrm{e}^x \Rightarrow f'(x) = \mathrm{e}^x$
	Derivative of $\ln x$	$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$
	Derivative of $\sec x$	$f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$
	Derivative of $\csc x$	$f(x) = \csc x \Rightarrow f'(x) = -\csc x \cot x$
	Derivative of $\cot x$	$f(x) = \cot x \Rightarrow f'(x) = -\csc^2 x$
	Derivative of a^x	$f(x) = a^x \Rightarrow f'(x) = a^x (\ln a)$
	Derivative of $\log_a x$	$f(x) = \log_a x \Rightarrow f'(x) = \frac{1}{x \ln a}$
	Derivative of $\arcsin x$	$f(x) = \arcsin x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$
	Derivative of $\arccos x$	$f(x) = \arccos x \Rightarrow f'(x) = -\frac{1}{\sqrt{1-x^2}}$
	Derivative of $\arctan x$	$f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1+x^2}$
7.2	Chain rule	$y = g(u)$, where $u = f(x) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$
	Product rule	$y = uv \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$
	Quotient rule	$y = \frac{u}{v} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Topic 7—Core: Calculus (continued)

7.4	Standard integrals	$\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$ $\int \frac{1}{x} \,\mathrm{d}x = \ln\lvert x\rvert + C$ $\int \sin x \,\mathrm{d}x = -\cos x + C$ $\int \cos x \,\mathrm{d}x = \sin x + C$ $\int \mathrm{e}^x \,\mathrm{d}x = \mathrm{e}^x + C$ $\int a^x \,\mathrm{d}x = \frac{1}{\ln a} a^x + C$ $\int \frac{1}{a^2 + x^2} \mathrm{d}x = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$ $\int \frac{1}{\sqrt{a^2 - x^2}} \mathrm{d}x = \arcsin\left(\frac{x}{a}\right) + C, \quad \lvert x\rvert < a$
7.5	Area under a curve Volume of revolution (rotation)	$A = \int_a^b y \mathrm{d}x$ or $A = \int_a^b x \mathrm{d}y$ $V = \int_a^b \pi y^2 \mathrm{d}x$ or $V = \int_a^b \pi x^2 \mathrm{d}y$
7.9	Integration by parts	$\int u \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \mathrm{d}x$ or $\int u \mathrm{d}v = uv - \int v \mathrm{d}u$

Topic 8—Option: Statistics and probability (further mathematics SL topic 2)

8.1 (2.1)	Linear combinations of two independent random variables X_1, X_2	$\mathrm{E}(a_1X_1 \pm a_2X_2) = a_1\mathrm{E}(X_1) \pm a_2\mathrm{E}(X_2)$ $\mathrm{Var}(a_1X_1 \pm a_2X_2) = a_1^2\,\mathrm{Var}(X_1) + a_2^2\,\mathrm{Var}(X_2)$
8.4 (2.4)	**Confidence intervals**	
	Mean, with known variance	$\bar{x} \pm z \times \frac{\sigma}{\sqrt{n}}$
	Mean, with unknown variance	$\bar{x} \pm t \times \frac{s_{n-1}}{\sqrt{n}}$
	Population	$\hat{P} \pm z\sqrt{\frac{\hat{P}(1-\hat{P})}{n}}$, where $\hat{P}$ is the proportion of successes in the sample
8.5 (2.5)	**Test statistics**	
	Mean, with known variance	$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$
	Mean, with unknown variance	$t = \frac{\bar{x} - \mu}{s_{n-1}/\sqrt{n}}$
8.6 (2.6)	The χ^2 test statistic	$\chi^2_{calc} = \sum \frac{(f_o - f_e)^2}{f_e} = \sum \frac{f_o^2}{f_e} - n$, where f_o are the observed frequencies, f_e are the expected frequencies, $n = \sum f_o$

Topic 9—Option: Sets, relations and groups (further mathematics SL topic 3)

9.1 (3.1)	De Morgan's laws	$(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$

Topic 10—Option: Series and differential equations (further mathematics SL topic 4)

10.5 (4.5)	Maclaurin series	$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$
	Taylor series	$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$
	Taylor approximations (with error term $R_n(x)$)	$f(x) = f(a) + (x-a) f'(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a) + R_n(x)$
	Lagrange form	$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, where c lies between a and x
	Integral form	$R_n(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, \mathrm{d}t$
	Other series	$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \ldots$ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$
10.6 (4.6)	Euler's method	$y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$, where h is a constant
	Integrating factor for $y' + P(x)y = Q(x)$	$\mathrm{e}^{\int P(x)\mathrm{d}x}$

Topic 11—Option: Discrete mathematics (further mathematics SL topic 5)

11.6 (5.6)	Euler's relation	$v - e + f = 2$, where v is the number of vertices, e is the number of edges, f is the number of faces
	Planar graphs	$e \le 3v - 6$ $e \le 2v - 4$

Formulae for distributions
(topic 8.2, further mathematics SL topic 2.2)

Discrete distributions

Distribution	Notation	Probability mass function	Mean	Variance
Bernoulli	$X \sim \mathrm{B}(1, p)$	$p^x(1-p)^{1-x}$ for $x = 0, 1$	p	$p(1-p)$
Binomial	$X \sim \mathrm{B}(n, p)$	$\binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0,1,\ldots,n$	np	$np(1-p)$
Hypergeometric	$X \sim \mathrm{Hyp}(n, M, N)$	$\dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$ for $x = 0,1,\ldots,n$	np where $p = \dfrac{M}{N}$	$np(1-p)\left(\dfrac{N-n}{N-1}\right)$ where $p = \dfrac{M}{N}$
Poisson	$X \sim \mathrm{P_o}(m)$	$\dfrac{m^x \mathrm{e}^{-m}}{x!}$ for $x = 0,1\ldots$	m	m
Geometric	$X \sim \mathrm{Geo}(p)$	pq^{x-1} for $x = 1, 2,\ldots$	$\dfrac{1}{p}$	$\dfrac{q}{p^2}$
Negative binomial	$X \sim \mathrm{NB}(r, p)$	$\binom{x-1}{r-1} p^r q^{x-r}$ for $x = r, r+1,\ldots$	$\dfrac{r}{p}$	$\dfrac{rq}{p^2}$
Discrete uniform	$X \sim \mathrm{DU}(n)$	$\dfrac{1}{n}$ for $x = 1,\ldots,n$	$\dfrac{n+1}{2}$	$\dfrac{n^2-1}{12}$

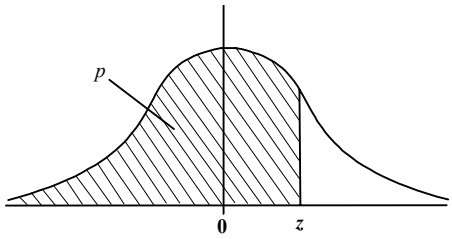
Continuous distributions

Distribution	Notation	Probability density function	Mean	Variance
Uniform	$X \sim \mathrm{U}(a,b)$	$\frac{1}{(b-a)},\ a \le x \le b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	$X \sim \mathrm{Exp}(\lambda)$	$\lambda \mathrm{e}^{-\lambda x},\ \ x \ge 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Normal	$X \sim \mathrm{N}(\mu,\sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	μ	σ^2

Statistical tables

Area under the standard normal curve (topic 6.11)

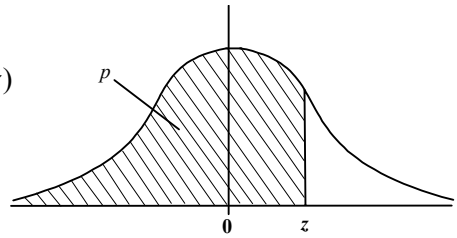
$p = \mathrm{P}(Z \leq z)$



z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8079	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9892	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9983	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9988	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998

Inverse normal probabilities (topic 6.11)

$p = \mathrm{P}(Z \leq z)$



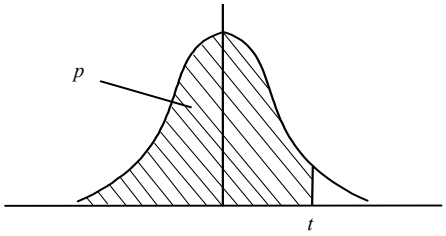
p	0	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.50	0.0000	0.0025	0.0050	0.0075	0.0100	0.0125	0.0150	0.0176	0.0201	0.0226
0.51	0.0251	0.0276	0.0301	0.0326	0.0351	0.0376	0.0401	0.0426	0.0451	0.0476
0.52	0.0502	0.0527	0.0552	0.0577	0.0602	0.0627	0.0652	0.0677	0.0702	0.0728
0.53	0.0753	0.0778	0.0803	0.0828	0.0853	0.0878	0.0904	0.0929	0.0954	0.0979
0.54	0.1004	0.1030	0.1055	0.1080	0.1105	0.1130	0.1156	0.1181	0.1206	0.1231
0.55	0.1257	0.1282	0.1307	0.1332	0.1358	0.1383	0.1408	0.1434	0.1459	0.1484
0.56	0.1510	0.1535	0.1560	0.1586	0.1611	0.1637	0.1662	0.1687	0.1713	0.1738
0.57	0.1764	0.1789	0.1815	0.1840	0.1866	0.1891	0.1917	0.1942	0.1968	0.1993
0.58	0.2019	0.2045	0.2070	0.2096	0.2121	0.2147	0.2173	0.2198	0.2224	0.2250
0.59	0.2275	0.2301	0.2327	0.2353	0.2379	0.2404	0.2430	0.2456	0.2482	0.2508
0.60	0.2534	0.2559	0.2585	0.2611	0.2637	0.2663	0.2689	0.2715	0.2741	0.2767
0.61	0.2793	0.2819	0.2845	0.2872	0.2898	0.2924	0.2950	0.2976	0.3002	0.3029
0.62	0.3055	0.3081	0.3107	0.3134	0.3160	0.3186	0.3213	0.3239	0.3266	0.3292
0.63	0.3319	0.3345	0.3372	0.3398	0.3425	0.3451	0.3478	0.3505	0.3531	0.3558
0.64	0.3585	0.3611	0.3638	0.3665	0.3692	0.3719	0.3745	0.3772	0.3799	0.3826
0.65	0.3853	0.3880	0.3907	0.3934	0.3961	0.3989	0.4016	0.4043	0.4070	0.4097
0.66	0.4125	0.4152	0.4179	0.4207	0.4234	0.4262	0.4289	0.4316	0.4344	0.4372
0.67	0.4399	0.4427	0.4454	0.4482	0.4510	0.4538	0.4565	0.4593	0.4621	0.4649
0.68	0.4677	0.4705	0.4733	0.4761	0.4789	0.4817	0.4845	0.4874	0.4902	0.4930
0.69	0.4959	0.4987	0.5015	0.5044	0.5072	0.5101	0.5129	0.5158	0.5187	0.5215
0.70	0.5244	0.5273	0.5302	0.5331	0.5359	0.5388	0.5417	0.5446	0.5476	0.5505
0.71	0.5534	0.5563	0.5592	0.5622	0.5651	0.5681	0.5710	0.5740	0.5769	0.5799
0.72	0.5828	0.5858	0.5888	0.5918	0.5948	0.5978	0.6008	0.6038	0.6068	0.6098
0.73	0.6128	0.6158	0.6189	0.6219	0.6250	0.6280	0.6311	0.6341	0.6372	0.6403
0.74	0.6434	0.6464	0.6495	0.6526	0.6557	0.6588	0.6620	0.6651	0.6682	0.6714
0.75	0.6745	0.6776	0.6808	0.6840	0.6871	0.6903	0.6935	0.6967	0.6999	0.7031

Inverse normal probabilities (topic 6.11, continued)

p	0	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.76	0.7063	0.7095	0.7128	0.7160	0.7192	0.7225	0.7257	0.7290	0.7323	0.7356
0.77	0.7389	0.7421	0.7455	0.7488	0.7521	0.7554	0.7588	0.7621	0.7655	0.7688
0.78	0.7722	0.7756	0.7790	0.7824	0.7858	0.7892	0.7926	0.7961	0.7995	0.8030
0.79	0.8064	0.8099	0.8134	0.8169	0.8204	0.8239	0.8274	0.8310	0.8345	0.8381
0.80	0.8416	0.8452	0.8488	0.8524	0.8560	0.8596	0.8633	0.8669	0.8706	0.8742
0.81	0.8779	0.8816	0.8853	0.8890	0.8927	0.8965	0.9002	0.9040	0.9078	0.9116
0.82	0.9154	0.9192	0.9230	0.9269	0.9307	0.9346	0.9385	0.9424	0.9463	0.9502
0.83	0.9542	0.9581	0.9621	0.9661	0.9701	0.9741	0.9782	0.9822	0.9863	0.9904
0.84	0.9945	0.9986	1.0027	1.0069	1.0110	1.0152	1.0194	1.0237	1.0279	1.0322
0.85	1.0364	1.0407	1.0451	1.0494	1.0537	1.0581	1.0625	1.0669	1.0714	1.0758
0.86	1.0803	1.0848	1.0894	1.0939	1.0985	1.1031	1.1077	1.1123	1.1170	1.1217
0.87	1.1264	1.1311	1.1359	1.1407	1.1455	1.1504	1.1552	1.1601	1.1651	1.1700
0.88	1.1750	1.1800	1.1850	1.1901	1.1952	1.2004	1.2055	1.2107	1.2160	1.2212
0.89	1.2265	1.2319	1.2372	1.2426	1.2481	1.2536	1.2591	1.2646	1.2702	1.2759
0.90	1.2816	1.2873	1.2930	1.2988	1.3047	1.3106	1.3165	1.3225	1.3285	1.3346
0.91	1.3408	1.3469	1.3532	1.3595	1.3658	1.3722	1.3787	1.3852	1.3917	1.3984
0.92	1.4051	1.4118	1.4187	1.4255	1.4325	1.4395	1.4466	1.4538	1.4611	1.4684
0.93	1.4758	1.4833	1.4909	1.4985	1.5063	1.5141	1.5220	1.5301	1.5382	1.5464
0.94	1.5548	1.5632	1.5718	1.5805	1.5893	1.5982	1.6073	1.6164	1.6258	1.6352
0.95	1.6449	1.6546	1.6646	1.6747	1.6849	1.6954	1.7060	1.7169	1.7279	1.7392
0.96	1.7507	1.7624	1.7744	1.7866	1.7991	1.8119	1.8250	1.8384	1.8522	1.8663
0.97	1.8808	1.8957	1.9110	1.9268	1.9431	1.9600	1.9774	1.9954	2.0141	2.0335
0.98	2.0538	2.0749	2.0969	2.1201	2.1444	2.1701	2.1973	2.2262	2.2571	2.2904
0.99	2.3264	2.3656	2.4089	2.4573	2.5121	2.5758	2.6521	2.7478	2.8782	3.0902

Critical values of the student's *t*-distribution (topic 8.4, further mathematics SL topic 2.4)

$p = \mathrm{P}(X \leq t)$

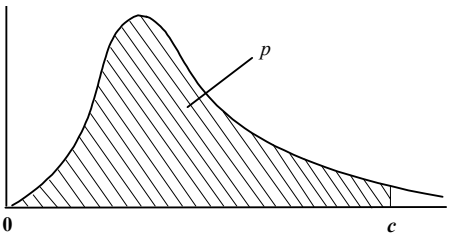


p	0.9	0.95	0.975	0.99	0.995	0.9995
$v = 1$	3.078	6.314	12.706	31.821	63.657	636.619
2	1.886	2.920	4.303	6.965	9.925	31.599
3	1.638	2.353	3.182	4.541	5.841	12.924
4	1.533	2.132	2.776	3.747	4.604	8.610
5	1.476	2.015	2.571	3.365	4.032	6.869
6	1.440	1.943	2.447	3.143	3.707	5.959
7	1.415	1.895	2.365	2.998	3.499	5.408
8	1.397	1.860	2.306	2.896	3.355	5.041
9	1.383	1.833	2.262	2.821	3.250	4.781
10	1.372	1.812	2.228	2.764	3.169	4.587
11	1.363	1.796	2.201	2.718	3.106	4.437
12	1.356	1.782	2.179	2.681	3.055	4.318
13	1.350	1.771	2.160	2.650	3.012	4.221
14	1.345	1.761	2.145	2.624	2.977	4.140
15	1.341	1.753	2.131	2.602	2.947	4.073
16	1.337	1.746	2.120	2.583	2.921	4.015
17	1.333	1.740	2.110	2.567	2.898	3.965
18	1.330	1.734	2.101	2.552	2.878	3.922
19	1.328	1.729	2.093	2.539	2.861	3.883
20	1.325	1.725	2.086	2.528	2.845	3.850
21	1.323	1.721	2.080	2.518	2.831	3.819
22	1.321	1.717	2.074	2.508	2.819	3.792
23	1.319	1.714	2.069	2.500	2.807	3.768
24	1.318	1.711	2.064	2.492	2.797	3.745
25	1.316	1.708	2.060	2.485	2.787	3.725
26	1.315	1.706	2.056	2.479	2.779	3.707
27	1.314	1.703	2.052	2.473	2.771	3.690
28	1.313	1.701	2.048	2.467	2.763	3.674
29	1.311	1.699	2.045	2.462	2.756	3.659
30	1.310	1.697	2.042	2.457	2.750	3.646
40	1.303	1.684	2.021	2.423	2.704	3.551
60	1.296	1.671	2.000	2.390	2.660	3.460
120	1.289	1.658	1.980	2.358	2.617	3.373
***	1.282	1.645	1.960	2.326	2.576	3.291

v = number of degrees of freedom

Critical values of the χ^2-distribution (topic 8.6, further mathematics SL topic 2.6)

$p = \mathrm{P}(X \leq c)$



p	0.005	0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99	0.995
$\nu = 1$	0.00004	0.0002	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169

ν = number of degrees of freedom